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Exam for the Course

Scientific Computing

2023/2024 – Exam – 13:45-16:45

You are allowed to bring a formula sheet (of standard size, format DIN A4, i.e., of size 21,0cm*29,7cm or less) with you. You may write on both sides of the formula sheet. The use of any further notes or electronic devices is not allowed. All of your answers need to be justified. Good luck!

Exercise 1 (LU/Cholesky factorization) (3 + 2 +1 Points)

We have given the following matrix A:

$$A = \begin{pmatrix} 4 & -4 & 8 \\ -4 & 5 & -8 \\ 8 & -8 & 17 \end{pmatrix}.$$

a) Compute the LU factorization of A <u>without</u> pivoting, where L has to be a unit lower triangular matrix. Use the LU factorization to compute the solution of $Ax = b$ for $b = (4, -2, -9)^T$.

b) Compute the Cholesky factorization of A.

c) How are the factorizations of part a) and b) related?

Exercise 2 (SVD) (2+4 Points)

a) Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$ and let $A = U\Sigma V^T$ be the singular value decomposition of A with $U \in \mathbb{R}^{m\times m}, \Sigma \in \mathbb{R}^{m\times n}, V \in \mathbb{R}^{n\times n}$. Show that for the singular values $\sigma_i(A), i = 1, \ldots, n$, it holds that

$$\sigma_i(A) = \sqrt{\lambda_i\left(A^TA\right)} = \sqrt{\lambda_i\left(AA^T\right)}, \qquad i = 1, \ldots, n,$$

where $\lambda_i\left(A^TA\right), \lambda_i\left(AA^T\right), i = 1, \ldots, n$ are the eigenvalues of the respective matrix and the additional $m - n$ zero eigenvalues of AA^T in case $m > n$ are neglected.

b) Compute the <u>thin</u> singular value decomposition for

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}.$$

Exercise 3 (Conjugate Gradient) (1+4 Points)

We have given a vector $b = (1,1)^T$ and a matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

a) What the requirements on the matrix A that the conjugate gradient method converges? Check if they are satisfied.

b) Compute the solution of this linear system with the conjugate gradient method and use $x_0 = (1/3, 2/3)^T$ as initial condition.

Exercise 4 (Céa Lemma) (2 Points)

Let X be a Hilbert space and let the bilinear form $b : X \times X \to \mathbb{R}$ be symmetric, continuous (with continuity constant C_B) and X-elliptic (with constant C_E). Additionally, assume that for any subspace X_h of X we have $y \in X$ and $y_h \in X_h$ such that

$$b(y, v) = g(v) \qquad \forall v \in X,$$
$$b(y_h, v_h) = g(v_h) \qquad \forall v_h \in X_h.$$

Prove the following estimate (also known as the Céa lemma)

$$\|y - y_h\|_X \leq \frac{C_B}{C_E} \inf_{v_h \in X_h} \|y - v_h\|_X .$$

Exercise 5 (Model order reduction) (2+1 Points)

Let X be a Hilbert space and let $b : X \times X \times \mathcal{P} \to \mathbb{R}$, $g : X \times \mathcal{P} \to \mathbb{R}$ be a parametric bilinear form/linear form with compact parameter domain $\mathcal{P} \subset \mathbb{R}^{n_p}$. Consider the parametric problem $P^{\mathrm{PDE}}(\mu)$ given by: Find $y(\mu) \in X$ such that,

$$b(y(\mu), v; \mu) = g(v; \mu), \qquad \forall v \in X.$$

b) How can the parametric bilinear form b be expanded, when the corresponding problem $P^{\mathrm{PDE}}(\mu)$ is parameter-separable? Why is the parameter-separability used for in the model order reduction process?

c) What is meant by the term "snapshot" in model order reduction?

Exercise 6 (Gershgorin circles) (3 + 2 Points)

a) Consider the matrix

$$A := \begin{pmatrix} 7 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}, \epsilon \in \mathbb{R}, |\epsilon| < 1.$$

Use the Gershgorin circles to approximately locate the eigenvalues of A and illustrate your findings in a picture. Show that for the smallest eigenvalue $\lambda_{\min}$ it holds that $|\lambda_{\min} - 1| \leq \epsilon$.

b) Consider a matrix $A \in \mathbb{R}^{n \times n}$. Assume that the n Gershgorin circles (we only consider row circles here) are disjoint. Show, that all eigenvalues of A must be real.

Exercise 7 (PageRank) (3 Points)

Denote with $w_1, \ldots, w_n$ all webpages in the considered network and with $PR(w_i)$ the PageRank of page $w_i, i = 1, \ldots, n$. We denote by the vector $\mathbf{x} := (PR(w_1), \ldots, PR(w_n))^T$ the vector of all PageRanks. Then, we may formulate the following system of equations to compute $\mathbf{x}$

$$\mathbf{x} = \left[d(A+R) + ee^T \frac{1-d}{n} \right] \mathbf{x}, \qquad \text{where } e = (1, \ldots, 1)^T \in \mathbb{R}^n, d \in (0,1), \qquad (1)$$

$$a_{ij} = \begin{cases} 1/(C(w_j)) & \text{if } w_j \text{ points to } w_i, \\ 0 & \text{else} \end{cases} \quad \text{and } R_{ij} = \begin{cases} 1/n & \text{if } C(w_j) = 0. \\ 0 & \text{else} \end{cases}$$

Here, we denote by $C(w_j)$ we denote the number of links going out of page w_j. The Simplified PageRank algorithm just considers the following system of equations:

$$\mathbf{x} = A\mathbf{x}.$$

Describe what two problems can occur with the Simplified PageRank algorithm and how these problems are solved with the PageRank algorithm (1). Illustrate your discussion with small graphs. (Hint: Consider the properties of a stochastic matrix.)

Exercise 8 (Mixed-integer Programming Model) (3 +2+1 Points)

We consider a robot that moves in the plane, starting at a place s and stopping at place t. It has to visit a (finite) set of places V, e.g., to water plants, in any order. For simplicity we assume $s, t \in V$. The time $d_{u,v} > 0$ (in seconds) that the robot needs to move from u to v is given for all $u, v \in V$ with $u \neq v$.

(a) Create a mixed-integer *linear* program to determine an optimal ordering of the places V such that the robot arrives at t as early as possible (having visited all places in $V \setminus \{t\}$, starting at s)! Briefly explain the meaning of your variables and constraints (one phrase or sentence each)!

(b) Generalize the model to a mixed-integer *nonlinear* program in order to incorporate the following adaptation: when visiting places $u, v, w \in V$ in that order directly after another then the robot needs to turn, which takes $r_{u,v,w} \geq 0$ seconds (in addition to $d_{u,v} + d_{v,w}$ for the movement). Adapt the objective function accordingly to take this turning time into account!

(c) Linearize the model in order to again obtain a mixed-integer *linear* program that also models the generalized problem.

Exercise	1.	2.	3.	4.	5.	6.	7.	8.	total	Grade:
Points	3+2+1	2+4	1+4	2	2+1	3+2	3	3+2+1	36	(Points + 4)/4