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Exam for the Course

Scientific Computing

2024/2025 – Exam – 13:45-16:45

You are allowed to bring a formula sheet (of standard size, format DIN A4, i.e., of size 21,0cm*29,7cm or less) with you. You may write on both sides of the formula sheet. The use of any further notes or electronic devices is not allowed. All of your answers need to be justified. Good luck!

Exercise 1 (QR factorization) (2+2+3 Points)

We are given the following matrix A and vector b:

$$A = \begin{pmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 7 \\ 7 \\ 14 \end{pmatrix}$$

a) Compute a reduced QR factorization of A using the Gram-Schmidt algorithm.[1]

b) Compute the minimizer $x \in \mathbb{R}^2$ of the problem

$$\min_{x \in \mathbb{R}^2} \|Ax - b\|_2$$

using the reduced QR decomposition calculated in a).

c) Here, we are given a non-singular matrix $A \in \mathbb{R}^{n \times n}$. Show (without arguing over the Gram-Schmidt algorithm) that the QR factorization is unique, provided that the diagonal entries r_{ii} of $R \in \mathbb{R}^{n \times n}$ are strictly positive, i.e., $r_{ii} > 0, i = 1, \dots, n$.

Exercise 2 (SVD) (3+3 Points)

a) Show that if $A \in \mathbb{R}^{m \times n}$ has rank m, then

$$\|A(A^T A)^{-1} A^T\|_2 = 1.$$

b) Compute the singular value decomposition for

$$A = \begin{pmatrix} 5 & 2 \\ 0 & 6 \end{pmatrix}.$$

[1] You are also allowed to compute the reduced QR decomposition via the Householder reflection.

Exercise 3 (Steepest descent/Conjugate gradient) (2+2 Points)
We have given a vector $b = (1,1)^T$ and a matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Moreover, we use $x_0 = (1/3, 2/3)^T$ as initial condition.

a) Given the linear system $Ax = b$ with initial condition x_0, does the steepest descent method converge? Does the same hold for the conjugate gradient method? Explain your answer.

b) Compute the solution of the linear system $Ax = b$ with initial condition x_0 with the steepest descent method until you arrive at x_1, i.e., x_1 is the last value that needs to be calculated.

Exercise 4 (Weak derivative) (2 Points)
Determine whether the following function has a weak derivative and if it does compute it. Explain your answer in full detail.

$$g(x) = \begin{cases} \frac{1}{2\sqrt{2}}x + \frac{3}{2\sqrt{2}} & \text{if } 0 \leq x \leq 1, \\ \sqrt{1+x^2} & \text{if } 1 < x \leq 2. \end{cases}$$

Exercise 5 (Model order reduction) (2+1 Points)

a) Describe the two phases in which the computations are separated in model order reduction to achieve efficient computations for a multi-query setting. What is the key difference between these two phases?

b) Where is the singular value decomposition used within model reduction and why?

Exercise 6 (Gershgorin circles) (2+2+2 Points)

a) Consider the matrix

$$A = \begin{pmatrix} 6 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 8 \end{pmatrix}.$$

Use the Gershgorin circles to approximately locate the eigenvalues of A and illustrate your findings in a picture.

b) Use the Gershgorin circles to determine an upper bound for the condition number of the matrix A with respect to the $\|\cdot\|_2$-norm. Recall to that end that $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$.

c) A matrix $A \in \mathbb{R}^{n\times n}$ is called *strictly diagonally dominant* if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad \text{for } i = 1, \ldots, n.$$

Show, using the Gershgorin circles, that every strictly diagonally dominant matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular.

Exercise 7 (PageRank) (1+1 Points)

a) Give the definition of a column-stochastic matrix.

b) Show that every column-stochastic matrix has 1 as an eigenvalue

Exercise 8 (Mixed-integer (non)linear programming model) (6 Points)
Philips wants to process a set T of tasks on m distinct (but identical) machines. Each task $t \in T$ can be processed on any machine, which takes a (positive) processing time of $p_t \in \mathbb{Z}$, independent of the choice of the machine. However, there is a deadline $d \in \mathbb{Z}$ (relative to the current time 0). Hence, it may not be possible to schedule all tasks within the time horizon $[0, d]$. However, they at least want to schedule as many tasks as possible to be completed.

(a) Create a mixed-integer *linear* program to determine an assignment of a largest-possible subset of the tasks T to the m machines such that each machine can process all its assigned tasks until the deadline d. Briefly explain the meaning of your variables and constraints (one phrase or sentence each)!

(b) Generalize the model to a mixed-integer *nonlinear* program in order to incorporate the following adaptation: For any pair (t_1, t_2) of (distinct) tasks there is a penalty value $v_{t_1,t_2} > 0$ that is subtracted from the objective in case both tasks, t_1 and t_2, are assigned to it. Adapt the objective function accordingly to take these penalties into account, that is, the new objective shall be the total number of jobs minus the total penalties.

(c) Linearize the model in order to again obtain a mixed-integer *linear* program that also models the generalized problem.

Exercise	1.	2.	3.	4.	5.	6.	7.	8.	total	Grade:
Points	2+2+3	3+3	2+2	2	2+1	2+2+2	1+1	6	36	(Points + 4)/4