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Exam for the Course

Scientific Computing

2025/2026 – Exam – 08:45-11:45

You are allowed to bring a formula sheet (of standard size, format DIN A4, i.e., of size 21,0cm*29,7cm or less) with you. You may write on both sides of the formula sheet. The use of any further notes or electronic devices is not allowed. All of your answers need to be justified. Good luck!

Exercise 1 (QR factorization) (4+2 Points)

We are given the following matrix A

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

a) Compute a QR factorization of A using the Gram-Schmidt algorithm.

b) Let $A \in \mathbb{R}^{n \times n}$, with full rank and n being an even number. Additionally, A has the property that columns 1,3,5,...,$n-1$ are orthogonal to columns 2,4,6,...,n. In a QR factorization of A, what structure does R possess? Explain this structure.

Exercise 2 (SVD) (3+1 Points)

a) Compute the singular value decomposition for

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Let A be the matrix of 2a). What is the error of the best rank-1 approximation of A in the $\|\cdot\|_2$-norm? I.e., determine

$$\inf_{\substack{B \in \mathbb{R}^{3 \times 3} \\ \operatorname{rank}(B) \le 1}} \|A - B\|_2.$$

Exercise 3 (Weak derivative, bilinear forms) (2+3 Points)

a) Determine whether the following function has a weak derivative and if it does compute it. Explain your answer in full detail.

$$g(x) = \begin{cases} \frac{1}{2\sqrt{2}}x + \frac{3}{2\sqrt{2}} & \text{if } 0 \leq x \leq 1, \\ e^{(x-1)} & \text{if } 1 < x \leq 2. \end{cases}$$

b) Let $I := (a, b)$ and let $X = H_0^1(I) := \{v \in H^1(I) : v(a) = v(b) = 0\}$. Consider the bilinear form $b : X \times X \to \mathbb{R}$, with

$$b(y, v) := \int_I p(x)y'(x)v'(x) + q(x)y(x)v(x)\,dx,$$

with $q, p \in C^0(\bar{I})$ and $p(x) \geq p_0 > 0$, $q(x) \geq q_0 > 0$. Show that this bilinear form is symmetric, continuous and coercive.

Exercise 4 (Model order reduction) (2+1 Points)

Let X be a Hilbert space and let $b : X \times X \times \mathcal{P} \to \mathbb{R}$, $g : X \times \mathcal{P} \to \mathbb{R}$ be a parametric bilinear form/linear form with compact parameter domain $\mathcal{P} \subset \mathbb{R}^{n_p}$. Consider the parametric problem $P^{\text{PDE}}(\mu)$ given by: Find $y(\mu) \in X$ such that,

$$b(y(\mu), v; \mu) = g(v; \mu), \qquad \forall v \in X.$$

a) How can the parametric bilinear form b be expressed, when the corresponding problem $P^{\text{PDE}}(\mu)$ is parameter-separable? What is the parameter-separability used for in the model order reduction process?

b) What is meant by the term "snapshot" in model order reduction?

Exercise 5 (Gershgorin circles) (1+2+2 Points)

Our goal is to find out if matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

is invertible with the help of the Gershgorin circles.

a) Show that with applying the Gershgorin circles directly to matrix A, that we are unable to decide if A is invertible or not.

b) Consider the diagonal matrix $D = \text{diag}(1, t, 1)$ with $t > 0$ and the matrix $C = D^{-1}AD$. Use the Gershgorin circles of C to find the values of t such that C is invertible.

c) Prove that if two matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$ are similar, that they have the same eigenvalues. Based on this explain why the invertibility of C guarantees invertibility of A.

Exercise 6 (PageRank) (1+ 3 Points)

a) Show that every column-stochastic matrix has 1 as an eigenvalue.

b) Show that every column-stochastic matrix has a spectral radius of 1.

Exercise 7 (Randomized SVD) (3 Points)
Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$ with $\operatorname{rank}(A) = r$ and let $A = Q_r\Sigma_r V_r^\top$ be an SVD of A with $Q_r \in \mathbb{R}^{m\times r}$, $\Sigma_r \in \mathbb{R}^{r\times r}$ and $V_r \in \mathbb{R}^{n\times r}$. Let $\Omega \in \mathbb{R}^{n\times \ell}$ with $\ell \geq r$ and set $Y = A\Omega \in \mathbb{R}^{m\times \ell}$. In addition, let U be a matrix with orthonormal columns such that $\operatorname{range}(U) = \operatorname{range}(Y)$. Prove that the randomized SVD recovers an exact deterministic SVD of A, if $\operatorname{rank}(V_r^\top\Omega) = r$. (Hint: first show that $A = UU^\top A$.)

Exercise 8 (Mixed-integer Linear Programming Model) (4.5+1+0.5 Points)
A university operates a fleet of whiteboard maintenance drones, specified by the set D. Every morning, the drones leave the maintenance depot with a stock of whiteboard pens. Across campus there are rooms (set R) whose pens must be replaced before lectures begin. Every room $r \in R$ requires exactly one service during which $p_r \in \mathbb{Z}$ pens are exchanged. Each drone starts at the maintenance depot, may visit several rooms, and eventually returns to the depot, denoted by O. It may also visit no room at all. Since drones must be small, each of them can carry only a limited number of pens, say, $Q_d \in \mathbb{Z}$ for drone $d \in D$.
The travel time $c_{i,j} > 0$ between every pair of locations $i, j \in R \cup \{O\}$ is known. The objective is to minimize the total travel time while ensuring that every such room is serviced exactly once.

(a) Create a mixed-integer *linear* program to determine an assignment of drones to rooms and ordering them for a visit such that the total travel time is minimized. Briefly explain the meaning of your variables and constraints (one phrase or sentence each)!

(b) Adapt the model to a mixed-integer *nonlinear* program in order to incorporate the following adaptation: Instead of the sum of travel times, the maximum over all drones shall be minimized. To this end, adapt the objective function accordingly to take this into account, that is, aim for the earliest moment at which all drones returned.

(c) Linearize the model in order to again obtain a mixed-integer *linear* program that also models the adapted version. Note that you may need to add (an) extra variable(s).

Exercise	1.	2.	3.	4.	5.	6.	7.	8.	total	Grade:
Points	4+2	3+1	2+3	2+1	1+2+2	1+3	3	4.5+1+0.5	36	(Points + 4)/4