191531750 Stochastic Processes – Part 2
Exam. Date: 02-02-2017, 13:45-15:45

In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 4 problems. The total number of points is 17. Good luck!

Throughout this exam standard Brownian motion denotes the Brownian motion without drift, with unit variance, starting at 0*.*

Problem 1.
[4 pt] Consider the bounded martingale $\{M_n\}_{n=0}^{\infty}$. Let $X_n = \sum_{k=1}^{n} \frac{1}{k}(M_k - M_{k-1})$. Use the martingale convergence theorem to prove that $\{X_n\}_{n=0}^{\infty}$ converges with probability one. *(Hint: Prove that* $\{X_n\}_{n=0}^{\infty}$ *is bounded.)*

Problem 2.
Let $\{X(t), t \geq 0\}$ be standard Brownian Motion.

a) [1 pt] Give the definition of standard Brownian motion

b) [3 pt] Use the reflection principle to derive an expression for $P(\max_{0\leq u\leq t} X(u) \leq x, a \leq X(t) < b)$.

Problem 3.
[3 pt] Let $\{X(t), t \geq 0\}$ be Brownian Motion with drift $\mu > 0$ and variance σ^2. Prove that $\{Y(t), t \geq 0\}$,

$$Y(t) = \frac{c}{\sigma} X\left(\frac{t}{c^2}\right) - \frac{\mu t}{\sigma c},$$

for any constant $c > 0$, is a standard Brownian Motion.

Problem 4.
We consider the price of a stock $\{S(t), t \geq 0\}$ as given by $S(t) = S_0 \exp(\mu t + \sigma X(t))$, where $\{X(t), t \geq 0\}$ is standard Brownian motion and $\sigma > 0$. Also, let $\{Z(t), t \geq 0\}$ denote the discounted stock price at time t in the presence of interest rate r, i.e. $Z(t) = \exp(-rt)S(t)$.

a) [3 pt] Show that if $\mu = r - \sigma^2/2$, the discounted stock price $\{Z(t), t \geq 0\}$ is a martingale.

b) [3 pt] Consider $K > 0$ and $T > 0$. Show that, again for $\mu = r - \sigma^2/2$,

$$\mathbb{E}\left[e^{-rT} \max\{S(T) - K, 0\}\right] = S_0 \Phi(\sigma\sqrt{T} - \alpha) - e^{-rT} K \Phi(-\alpha),$$

where

$$\alpha = \frac{\log\left(\frac{K}{S_0}\right) - \mu T}{\sigma\sqrt{T}}, \quad \Phi(x) = \int_{-\infty}^{x} 1/\sqrt{2\pi} \exp(-u^2/2) \mathrm{d}u.$$

Total: 17 points