191531750 Stochastic Processes
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In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 7 problems. The total number of points is 36.
Good luck!

1. Consider a renewal process, where $X_1, X_2, \dots$ are i.i.d. interarrival times with non-arithmetic distribution $F(\cdot)$. Let γ_t be the excess life of the process at time t and δ_t the age of the process at time t.

   [3 pt] For $x, s > 0$, express
   $$\mathbb{P}(\gamma_t > x \mid \delta_{t+x/2} = s), \tag{1}$$
   in terms of $F(\cdot)$. (You may assume that $t + x/2 > s$)

2. Consider a renewal process $N(t)$ with i.i.d. inter-renewal times $X_1, X_2, \dots$ Suppose that the interarrival distribution F is non-arithmetic with mean $0 < \mu < \infty$.

   a) (3 pt) Show that
   $$Z(t) = \mathbb{P}(\gamma_t > x, \delta_t > y) = (1 - F(t+x))I_{t \in (y,\infty)} + \int_0^t \mathbb{P}(\gamma_{t-x} > x, \delta_{t-x} > y)dF(x)$$

   b) (3 pt) Calculate $\lim_{t\to\infty} Z(t)$.

3. (4 pt) The life of a car is a random variable with distribution $F(x)$. The car owner Mr. Brown has a policy of trading in his car either when it fails or reaches the age of A years. Thus, over the years Mr. Brown owns several cars (though only one at the same time), each being immediately replaced by a new car once it is sold.

   Let $R(A)$ denote the resale value of an A-year old car. The resale value of a failed car is C_1. Let C_2 denote the cost of a new car and suppose that an additional cost C_3 is incurred whenever the car fails. Give a formula for the long-run average cost per unit time in terms of F and A and the given cost constants.

4. Let $\{X_n\}_{n\geq 0}$ be i.i.d. with $\mathbb{P}(X_i = 1) = p$, $\mathbb{P}(X_i = -1) = q$, $\mathbb{P}(X_i = 0) = r$ and $X_0 = 0$, where $p + q + r = 1$ and $p > q > 0$. Let $S_n := X_1 + X_2 + \cdots + X_n$.

   a)[3 pt] For what values of $C \neq 1$ will $M_n := C^{S_n}$ be a martingale?

   b)[3 pt] Let $T_k = \min\{i : S_i = k\}$. For $a, b > 0$, compute
   $$\mathbb{P}(T_{-a} < T_b).$$

5. Consider a bounded martingale $M_n, n \geq 0$, where $|M_n| \leq K$ for all n and some $K > 0$. Let $X_n = \sum_{k=1}^{n} \frac{1}{k}(M_k - M_{k-1})$.

   a) [2 pt] Show that for $n \geq 1$,

$$X_n = \frac{M_n}{n} - M_0 + \sum_{k=1}^{n-1} \frac{M_k}{k(k+1)}. \tag{2}$$

   b)[3 pt] Use the martingale convergence theorem to prove that X_n converges with probability one. *(Hint: Prove that X_n is a bounded martingale. You may use that $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} < \infty$.)*

6. a) [2 pt] Give the definition of Brownian Motion

   b) [3pt] Is $C(t) = B(2t) - B(t)$ a Brownian Motion when $B(t)$ is a Brownian Motion?

   c) [3 pt] Show that for a standard Brownian Motion $A(t)$ and for $k > 0$,

$$\mathbb{E}[|A(t)|^k] = \frac{(2t)^{k/2}}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right),$$

   where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$.

7. (4 pt) Let $\{X(t), t \geq 0\}$ be a standard Brownian motion and define $\{Y(t), t \in [0,1]\}$ as

$$Y(t) = X(t) - tX(1),$$

   for $t \in [0,1]$. Moreover, define $\{Z(t), t \in [0,1]\}$ as

$$Z(t) = (1+t)Y\left(\frac{t}{1+t}\right),$$

   for $t \in [0,1]$.

   Prove that $\{Z(t), t \in [0,1]\}$ is a Brownian Motion.