191531750 Stochastic Processes
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In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 7 problems. The total number of points is 36.
Good luck!

1. Consider a renewal process, where $X_1, X_2, \ldots$ are i.i.d. interarrival times with non-arithmetic distribution $F(\cdot)$. Let γ_t be the excess life of the process at time t and δ_t the age of the process at time t.

   [3 pt] For a Poisson Process with intensity λ and $x, s > 0$, compute

$$\mathbb{P}(\gamma_t > x \mid \delta_{t+x} > s). \tag{1}$$

2. (3 pt) Particles arrive at a counter according to a Poisson process with intensity λ. The jth particle locks the counter for a length Y_j of time, and cancels all locks of its predecessors. $Y_1, Y_2, \ldots$ are i.i.d. with distribution G and independent of the Poisson process. The first particle arrives at the counter at time 0. Let L be the length of the first time interval during which the counter is locked. Show that $H(t) = \mathbb{P}(L > t)$ satisfies

$$\mathrm{e}^{-\lambda t}[1 - G(t)] + \int_0^t H(t-x)[1-G(x)]\lambda \mathrm{e}^{-\lambda x}\mathrm{d}x$$

3. Buses arrive at a bus stop according to a renewal process with i.i.d. inter-arrival times $X_1, X_2, \ldots$ (in minutes), with $\mathbb{E}[X_1] < \infty$ and $\mathbb{E}[X_1^2] < \infty$. Some buses are green, the others are red. The probability that an arbitrary bus is red equals $p > 0$. The color of a bus is independent of the color of the other buses and independent of $X_1, X_2, \ldots$.

   a)[3 pt] People arrive with a rate of 1 person per minute to the bus stop. When a bus arrives, all waiting passengers pay €$1/p$ if the bus is red and €$1/(1-p)$ if the bus is green. What is the average long-run income rate of the bus company? (euro per time unit)

   b) [4pt] Let $A(t)$ denote the time that went by since the last bus arrived and $B(t)$ the time until the next bus arrives at time t. Let $J(t) = \int_0^t \min(A(s), B(s))ds$. By using an appropriate renewal reward process, show that

$$\lim_{t\to\infty} J(t)/t = \frac{\mathbb{E}[X_1^2]}{2\mathbb{E}[X_1]}.$$

4. Let $X_1, \ldots, X_n$ be i.i.d. with $\mathbb{P}(X_n = -1) = \mathbb{P}(X_n = 1) = \frac{1}{2}$. Set $S_n = X_0 + \cdots + X_n$.

   a) (4 pt) Show that $M_n = S_n^3 - 3nS_n$, is a martingale with respect to X_n.

   b) [4pt] Fix $m > 0$, and let T be the first time that the walk hits either 0 or m. Show that, for each $0 < k \leq m$, when $X_0 = k$,

$$\mathbb{E}[T \mid S_T = m] = \frac{m^2 - k^2}{3}.$$

   You may assume that the optimal stopping theorem can be applied to M_n.

5. Let $\{X_n\}_{n\geq 0}$ be independent random variables with $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = 1/2$. Furthermore, define $M_n = \sum_{i=1}^{n} X_i/i$.

   a) [3pt] Show that $\{M_n\}_{n\geq 0}$ is a martingale.

   b) [3pt] Use the Martingale Convergence Theorem to show that $\{M_n\}_{n\geq 0}$ converges with probability one and in mean square to a random variable M. You may use that $\sum_{i=1}^{\infty} 1/i^2 < \infty$.

6. a) [3 pt] Give the definition of Brownian Motion

   b) [3 pt] Use the reflection principle to derive an expression for $P(\max_{0\leq u\leq t} X(u) \leq x, a \leq X(t) < b)$.

   c) [3 pt] Let $A(t)$ and $B(t)$ be two independent standard Brownian Motions. Are there constants α, β such that $X(t) = \alpha A(t) + \beta B(t)$ is a standard Brownian motion again? If yes, find all conditions on α and β such that $X(t)$ is a standard Brownian Motion.

   d) [3 pt] Is $C(t) = Z\sqrt{t}$ a Brownian Motion when $Z \sim \mathcal{N}(0, 1)$?