191531750 Stochastic Processes
Exam. Date: 03-02-2012, 13:45-16:45

In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 4 problems. The total number of points is 36.

Good luck!

1. A lazy professor has a ceiling fixture in his office that contains two light bulbs. To replace a bulb, the professor must fetch a ladder, and being lazy, when a single bulb fails, he waits until the second bulb fails before replacing them both at the same time. Assume that the lengths of life of the bulbs are independent and exponentially distributed with parameter 2 (we choose one year as a time unit, so the average life time of a bulb is 1/2 year).

   a) [2pt] Give the definition of a renewal process.

   b) [3pt] What fraction of time, in the long run, is our professor's office half lit?

   c) [3pt] Provide an approximation for the average number of bulbs that the professor have used in n years, such that the difference between the exact average value and your approximation goes to zero as $n \to \infty$. Compute the approximation for the average number of bulbs used in 15 years.

   d) [2pt] Explain why a Poisson process with intensity λ is a stationary process.

2. Consider the Markov chain $\{Z_n\}_{n\geq 0}$ with state space $E = \{0, 1, \ldots, m\}$, $Z_0 = z_0$, and transition probabilities

$$p_{ij} = \begin{cases} 1, & i = j = 0 \text{ or } i = j = m \\ \binom{m}{j}\left(\frac{i}{m}\right)^j\left(1-\frac{i}{m}\right)^{m-j}, & \text{otherwise.} \end{cases}$$

   a) [3pt] Show that $\{Z_n\}_{n\geq 0}$ is a martingale.

   b) [3pt] Compute the probability of absorption by state 0.

   c) [2pt] Use the Martingale Convergence Theorem to show that $\{Z_n\}_{n\geq 0}$ converges with probability one to a random variable Z. Use b) to write down the distribution of Z.

3. Let X_i, $i = 1, 2, \ldots$ i.i.d. continuous random variables with $\mathbb{E}|X_i| < \infty$. Consider the random walk $S_0 = 0$, $S_n = \sum_{i=1}^n X_i$, $i = 1, 2, \ldots$.

   Let $N_1^a = \min\{n : S_n > 0\}$, $N_1^d = \min\{n : S_n < 0\}$, $A_1 = S_{N_1^a}$, $D_1 = S_{N_1^d}$, $A(t) = P(A_1 < t)$.

   Let N_i^a denote the time between the $(i-1)$st and ith ascending ladder variable, and N_i^d denote the time between the $(i-1)$st and ith descending ladder variable.

Z.O.Z.

Define the ascending and descending renewal processes with increments $N_i^a - N_{i-1}^a$, $i = 1, 2, \ldots$, and $N_i^d - N_{i-1}^d$, $i = 1, 2, \ldots$, respectively.

Let $m_a(t) = \sum_{n=1}^{\infty} A^{(n)}(t)$, where $A^{(n)}$ is the n-fold convolution of A.

a) [2pt] Argue that $m_a(t)$ is the expected number of ascending ladder heights in $(0, t]$.

b) [2pt] Give the definition of exchangeability.

c) [2pt] Write the expected number of visits of S_n to $(0, t]$ prior to its first visit to $(-\infty, 0]$ as

$$\sum_{n=0}^{\infty} \mathbb{P}(S_1 > 0, S_2 > 0, \ldots, S_n \in (0, t]),$$

and show that this number is equal to $m_a(t)$.

d) [2pt] Show that $\mathbb{E}N_1^d = 1 + m_a(\infty)$.

e) [2pt] Prove the following theorem.

Theorem. There exists only two types of random walks:
(i) *Oscillating.* $\{S_n\}$ oscillates w.p. 1 without bound between $-\infty$ and $+\infty$, and $\mathbb{E}N_1^a = \mathbb{E}N_1^d = \infty$.
(ii) *Drift.* $\{S_n\}$ drifts to either $+\infty$ (positive drift) or $-\infty$ (negative drift), e.g., for negative drift $S_n \to -\infty$ w.p. 1, and reaches a finite maximum. Either $\mathbb{E}N_1^a < \infty$ or $\mathbb{E}N_1^d < \infty$.

4. Let $B(t)$ be standard Brownian motion.

a) [1pt] Give the definition of a Gaussian process.

b) [2pt] Show that a process $\{X(t), t \geq 0\}$ is standard Brownian motion if and only if it is a Gaussian process with $\mathbb{E}X(t) = 0$ and Cov$(X(s), X(t)) = \min(s, t)$.

Let $B_1(t), \ldots, B_d(t)$ be standard Brownian motion, then $(B_1(t), \ldots, B_d(t))$ is d dimensional Brownian motion. Let

$$R(t) = \sqrt{(B_1(t))^2 + \cdots + (B_d(t))^2}$$

be its distance from the origin, and $S_r = \min\{t : R(t) = r\}$ be the first time the distance from the origin is equal to r.

c) [2pt] Show that $Z(t) = (R(t))^2 - td$ is a martingale.

d) [3pt] Show that

$$\mathbb{E}S_r = r^2/d.$$

In the proof assume that

$$\lim_{n\to\infty} \mathbb{E}[Z(\min\{n, S_r\})] = \mathbb{E}[Z(S_r)].$$

Total: 36 points