**191531750 Stochastic Processes**
**Exam. Date: 01-02-2013, 13:45-16:45**

In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 5 problems. The total number of points is 36. Good luck!

1. a) [2pt] Define a renewal equation and write it down in a general form.

   b) [3pt] Consider a renewal process where the times between events have a distribution $F$? that is not arithmetic with mean $\mu$. Let $\gamma_t$ be the excess time at $t > 0$. Derive $\lim_{t\to\infty} P(\gamma_t > x)$.

2. A lifetime $X$ of a piece of equipment has an exponential distribution $F(t) = P(X \leq t) = 1 - e^{-\lambda t}$. The equipment is replaced either upon a failure or upon reaching age $T$. The cost of a new equipment is $c_1$ euro. If the equipment is replaced upon a failure then there is an additional cost of $c_2$ euro and an additional random delay that has distribution function $G$ with mean $\mu$.

   a) [2pt] Name two possible renewal processes related to the replacement model above.

   b) [3pt] Derive the long-run average cost per unit time for this replacement model.

3. Let $\{X_n\}$ be a Markov chain with transition probabilities $P_{i,i+1} = p_i = 1 - P_{i,0}$ for $i = 0, 1, \ldots$. Suppose $0 < p_i < 1$ and $p_i \geq p_{i+1} \geq \cdots$. Fix $0 < \beta < 1$, and let $a$ be the unique value for which $a\beta p_{a-1}/(a-1) > 1 \geq (a+1)\beta p_a/a$. Define

$$f(i) = \begin{cases} a\beta^{a-i} p_i \cdot p_{i+1} \cdots p_{a-1}, & \text{for } i < a \\ i, & \text{for } i \geq a. \end{cases}$$

   a) [2pt] Show that $f(i) \geq i$ for all $i$.

   b) [2pt] Show that $f(i) \geq \beta E[f(X_n)|X_{n-1} = i]$ so that $\{\beta^n f(X_n)\}$ is a nonnegative supermartingale.

   c) [3pt] Let $T$ be a Markov time such that $P(T < \infty) = 1$. Using b), verify that $f(i) \geq E(\beta^T f(X_T)|X_0 = i)$, and then use a) to conclude that $f(i) \geq E(\beta^T X_T|X_0 = i)$ for all such Markov times.

   d) [3pt] Define $T^* = \min\{n \geq 0 : X_n \geq a\}$. Argue that $P(T^* < \infty) = 1$. Finally, prove that $f(i) = E[\beta^{T^*} X_{T^*}|X_0 = i]$. Thus, $T^*$ maximizes $E[\beta^T X_T|X_0 = i]$ over all Markov times $T$ such that $P(T < \infty) = 1$.

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4. Let $\{X_n\}$ denote a branching process, i.e. $X_0 = 1$ and $X_{n+1} = \sum_{r=1}^{X_n} Z_{n,r}$ for $n \geq 0$, where $Z_{n,r}$ are i.i.d. random variables with mean $\mu$. Let $Y_n = X_n/\mu^n$.

   a) [2pt] Show that $Y_n$ is a martingale.

   b) [3pt] Show that for any non-negative function $\lambda : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} \lambda(n) = +\infty$ holds

   $$\lim_{n\to\infty} P(\max_{0\leq k\leq n} Y_n > \lambda(n)) = 0.$$

   c) [3pt] Discuss the convergence of $\{X_n\}$. In particular, explain the influence of $\mu$.

5. Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. Define, for $a > 0$ and $b < 0$ , $T = \inf\{u \geq 0 : B(u) \in \{a, b\}\}$.

   a) [3pt] By applying the stopping theorem to the martingale $\{B(t), t \geq 0\}$ and the stopping time $T$ show that

   $$P(B(t) = a) = \frac{-b}{a-b}.$$

   Define now

   $$M(t) = \int_0^t B(u)du - \frac{1}{3}B(t)^3.$$

   b) [3pt] Show that $\{M(t), t \geq 0\}$ is a martingale.

   c) [2pt] Deduce that the expected area under the path of $B(t)$ until it first reaches one of the levels $a$ or $b$ is

   $$-\frac{1}{3}ab(a+b).$$

   Hint: apply once more the stopping theorem, this time to the martingale $\{M(t), t \geq 0\}$.

**Total:** 36 points