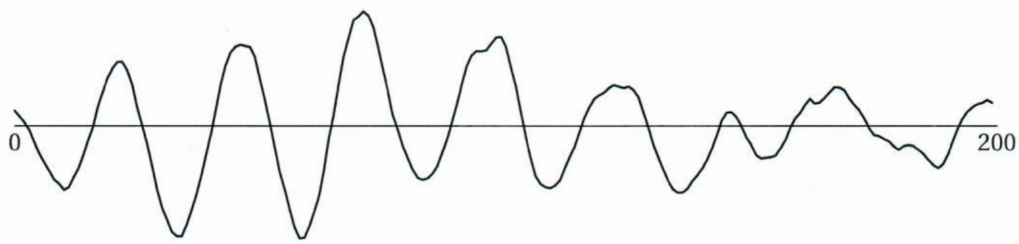
Time Series Analysis (& SI)—191571090

Date:	28-06-2019
Place:	CR-2K
Time:	08:45–11:45 (till 12:30 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	**The first 5 problems are CLOSED BOOK and WITHOUT CALCULATOR. Hand in the solution of these 5 before doing problems 6–8. For problems 6–8 use of lecture notes and calculator IS allowed.**



1. **CLOSED BOOK:** The above shows a realization of 200 samples generated with one the following three schemes. Explain which one it is:

   - $X_t = 0.95X_{t-1} + \epsilon_t$
   - $X_t = 1.9X_{t-1} - 0.95X_{t-2} + \epsilon_t$
   - $X_t = -1.9X_{t-1} - 0.95X_{t-2} + \epsilon_t$

2. **CLOSED BOOK:** In system identification what is an *ARX scheme*?

3. **CLOSED BOOK:** Consider the MA process

   $$X_t = (1 - \frac{1}{2}\mathrm{q}^{-2} + \frac{1}{16}\mathrm{q}^{-4})\epsilon_t$$

   and assume that $\mathbb{E}(\epsilon_t) = 0$ and $\mathrm{var}(\epsilon_t) = 1$.

   (a) Show that the scheme is invertible.

   (b) Compute the covariance function of X_t.

   (c) Determine the one-step ahead predictor $\hat{X}_{t|t-1}$ of X_t.

   (d) Is the two-step ahead predictor $\hat{X}_{t|t-2}$ of X_t the same as the one-step ahead predictor?

   (e) How does the one-step ahead predictor change if $\mathbb{E}(\epsilon_t) = 2$?

4. **CLOSED BOOK:** Suppose all we know about spectral densities is that (a) the average of $\phi_X(\omega)$ over $\omega \in [0,\pi]$ equals the power of X_t, and (b) that $\phi_X(\omega) = |H(\mathrm{e}^{\mathrm{i}\omega})|^2\phi_U(\omega)$ if $X_t = H(\mathrm{q})U_t$. Show that this alone already implies that $\phi_X(\omega)$ is real and nonnegative for all frequencies $\omega \in [0,\pi]$.

5. **CLOSED BOOK:** Suppose $X_0,\dots,X_{N-1}$ are independent and that they are uniformly distributed on $[0,\theta]$ for some $\theta > 0$.

   (a) Show that the joint density function is $f(x_1,\dots,x_N) = \begin{cases} \frac{1}{\theta^N} & \text{if all } x_i \in [0,\theta] \\ 0 & \text{elsewhere} \end{cases}$

   (b) Show that the maximum likelihood estimator of θ is $\hat{\theta} = \max_{t=0,\dots,N-1} X_t$.

6. **OPEN BOOK:** We consider estimation of AR models on the basis of $X_0,\dots,X_{N-1}$. Several applications exploit/assume the property that AR models $A(\mathrm{q})X_t=\epsilon_t$ computed with least squares make ϵ_t "close to white". We explore this for the case that

$$X_t=(1-b\,\mathrm{q}^{-1})\eta_t \qquad \text{with } |b|<1,$$

and η_t white noise with mean zero and $\operatorname{var}(\eta_t)=\sigma_\eta^2$.

(a) Suppose we try to model this X_t with an AR(1) scheme $X_t=aX_{t-1}+\epsilon_t$. Determine the a that minimizes

$$\mathbb{E}(\epsilon_t^2)$$

and show that $\mathbb{E}(\epsilon_t^2)\geq\sigma_\eta^2$.

(b) Now model this X_t with an AR(n) scheme $(1-a_1\mathrm{q}^{-1}-\cdots-a_n\mathrm{q}^{-n})X_t=\epsilon_t$. Show that $\mathbb{E}(\epsilon_t^2)\geq\sigma_\eta^2$ for every possible $a_1,\dots,a_n$.

(c) Next define $A_\infty(\mathrm{q})=\frac{1}{B(\mathrm{q})}=(1+b\,\mathrm{q}^{-1}+b^2\mathrm{q}^{-2}+b^3\mathrm{q}^{-3}+\cdots)$ and then truncate this to finite order n:

$$A_n(\mathrm{q})=1+b\,\mathrm{q}^{-1}+b^2\mathrm{q}^{-2}+\cdots+b^n\mathrm{q}^{-n}.$$

Let $A_n(\mathrm{q})X_t=\epsilon_t$. Show that $\lim_{n\to\infty}\mathbb{E}(\epsilon_t^2)=\sigma_\eta^2$.

(d) Argue that for the above $A_n(\mathrm{q})$ we have that "ϵ_t becomes more and more white as $n\to\infty$."

(e) Does the above easily imply that AR(n) models *found through standard least squares on* $X_0,\dots,X_{N-1}$ for the above X_t makes "ϵ_t become more and more white as $n\to\infty$ and $N:=100n\to\infty$"? [Keep your answer qualitative and short.]

7. **OPEN BOOK:** Consider

$$X_t=(1+b\,\mathrm{q}^{-1})\epsilon_t, \qquad b\in\mathbb{R},$$

with ϵ_t iid zero mean and normally distributed. How large does N need to be to ensure that the standard deviation of $\hat{\gamma}_N(0)$ (estimated from $X_0,\dots,X_{N-1}$) is at most 0.1 times $\gamma(0)$? [Your N must be valid for every possible b but N is not allowed to depend on b.]

8. **OPEN BOOK – System identification:** Consider $Y_t=\sum_{m=-\infty}^{\infty}h_mU_{t-m}+V_t$ and suppose U_t and V_t uncorrelated WSS processes.

(a) Prove that $\gamma_y(\tau)=\gamma_z(\tau)+\gamma_v(\tau)$ where $Z_t=\sum_{m=-\infty}^{\infty}h_mU_{t-m}$.

(b) If the coherence spectrum $K_{yu}(\omega)=1/4$ for all frequencies ω, which fraction of the power of y is due to v?

(c) If both U_t and V_t are white noise, can $|K_{yu}(\mathrm{e}^{\mathrm{i}\omega})|$ then exceed $|H(\mathrm{e}^{\mathrm{i}\omega})|$?

problem:	1	2	3	4	5	6	7	8
points:	2	2	2+2+2+1+2	2	2+2	2+2+2+2+2	2	2+1+2

Exam grade is $1+9p/p_{\max}$.