Time Series Analysis - Test 1

Coursecode: 191571090

Quartile: 2B 2021

Date: 01 July 2022

Time: 08:45 – 11:45

Teacher: Annika Betken

Student's name: [redacted]

Student number: [redacted]

General information:

- The test will be partly closed book (first half of the test, i.e. the first 90 minutes) and partly open book (second half of the test, i.e. the last 90 minutes). For the open book part you will get access to the lecture notes and lecture slides by means of Chromebooks.
- Other than a pen, no means are needed (or allowed) for answering the exam questions.
- Please write your name on every exam paper you hand in.
- Please write legibly. I cannot evaluate what I do not understand.
- Justify your answers. An answer without justification will not be given full points.

Distribution of points:

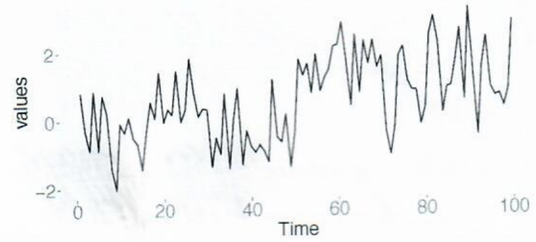
Exercise	1	2	3	4	5	6	7	Sum
Points	3+2+6	1+5	3+1+2+2+5	4+2	5+1+2+1+2	3+5	7	62

Exercise	1	2	3	4	5	6	7	Sum
Points								

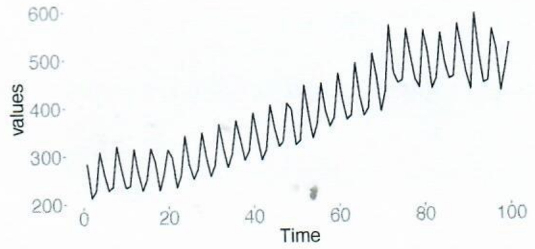
Hint: For solving the problems, you may make use of the fact that for any real number q, $\sum_{k=0}^{\infty} |q|^k < \infty$ if, and only if, $|q| < 1$.

1) **Closed book question**

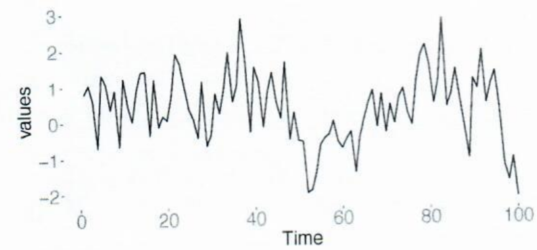
(a) Consider the time series plots (i)-(vi).



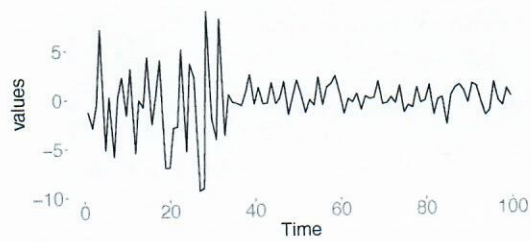
(i)



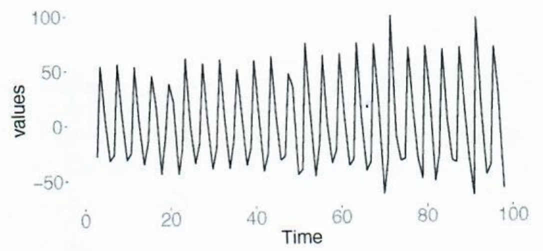
(ii)



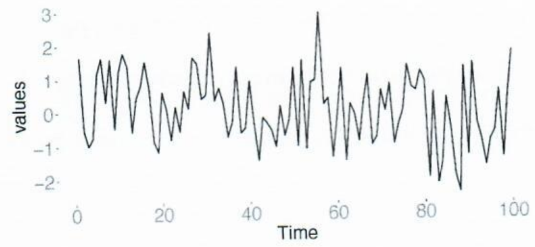
(iii)



(iv)



(v)



(vi)

Each of the time series is a realization of one of the following stochastic processes:

(1) A Gaussian white noise process.

(2) An AR(1) process with parameter $a = 0.7$.

(3) A stochastic process X_t, $t = 1, \dots, N$, (for some $N \in \mathbb{N}$) defined by

$$X_t = \begin{cases} \epsilon_t & t = 1, \dots, \lfloor \frac{N}{2} \rfloor \\ \epsilon_t + \mu & t = \lfloor \frac{N}{2} \rfloor + 1, \dots, N \end{cases}$$

for some $\mu > 0$ and a white noise process ϵ_t, $t = 1, \dots, N$. (For a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Question 1 continues on the next page. . .

(4) A stochastic process X_t, $t = 1, \dots, N$, (for some $N \in \mathbb{N}$) defined by

$$X_t = \begin{cases} \sigma\epsilon_t & t = 1, \dots, \lfloor \frac{N}{3} \rfloor \\ \epsilon_t & t = \lfloor \frac{N}{3} \rfloor + 1, \dots, N \end{cases}$$

for some $\sigma > 1$ and a white noise process ϵ_t, $t = 1, \dots, N$.

(5) The Australian beer production (in megalitres) recorded over a time period of 100 years.

(6) The detrended Australian beer production (in megalitres) recorded over a time period of 100 years.

Assign each of the plots (i)-(vi) the corresponding data generating stochastic process (1) - (6).

(b) Define wide-sense stationarity for a time series X_t, $t \in \mathbb{N}$.

(c) Which of the time series (i)-(vi) can be considered as wide-sense stationary? For each plot, justify your answer.

2) Closed book question

(a) What is meant by saying that a causal (stable) linear process is invertible?

(b) Let ϵ_t, $t \in \mathbb{N}$, be a white noise process. For each of the following time series models find an expression in terms of an ARMA difference equation and determine whether the model is invertible or not. For each model, justify your answer.

1. $X_t = 0.2X_{t-1} + \epsilon_t$
2. $X_t = \epsilon_t - 1.5\epsilon_{t-1} + 0.3\epsilon_{t-2}$
3. $X_t = 0.4X_{t-1} + \epsilon_t - 1.5\epsilon_{t-1} + 0.3\epsilon_{t-2}$

3) Closed book question

Consider the linear process $X_t = (1 + b\,\mathrm{q}^{-1})\epsilon_t$, where ϵ_t, $t \in \mathbb{N}$, is a white noise process with mean 0 and variance σ_ϵ^2, and q^{-1} denotes the backward shift operator, i.e. $\mathrm{q}^{-1} X_t = X_{t-1}$. Suppose that $|b| > 1$.

(a) Determine the autocovariance function $\gamma(k) := \mathrm{Cov}(X_1, X_{k+1})$ in terms of b and σ_ϵ^2.

(b) Define the spectral density of a wide-sense stationary process or (alternatively) give a formula for the spectral density of an ARMA process $\mathrm{A}(\mathrm{q})X_t = \mathrm{B}(\mathrm{q})\epsilon_t$.

(c) For the ARMA process defined by the scheme $(b + \mathrm{q}^{-1})Y_t = (1 + b\,\mathrm{q}^{-1})\epsilon_t$, where ϵ_t, $t \in \mathbb{N}$, is a white noise process with mean 0 and variance σ_ϵ^2, determine the spectral density.

(d) Argue that the given linear process $X_t = (1 + b\,\mathrm{q}^{-1})\epsilon_t$ can also be written as $X_t = (b + \mathrm{q}^{-1})\tilde{\epsilon}_t$ for some white noise process $\tilde{\epsilon}_t$, $t \in \mathbb{N}$.

(e) We could choose to estimate X_t as a function of X_{t-1}. Determine the constant $c \in \mathbb{R}$ that minimizes $\mathrm{E}\left((X_t - cX_{t-1})^2\right)$. For this constant c, compute the mean squared prediction error.

$$\left(1 - \frac{b^2}{(1+b^2)^2} \cdot (1+b^2)\right)\sigma_\varepsilon^2$$

4) Open book question

Again, consider the linear process $X_t = (1 + b\,\mathrm{q}^{-1})\epsilon_t$, where ϵ_t, $t \in \mathbb{N}$, is a white noise process with mean 0 and variance σ_ϵ^2. Suppose that $|b| > 1$.

(a) Determine the 1-step ahead predictor $\hat{X}_{t|t-1}$ and compute its mean squared prediction error.

(b) Compare this predictor to the predictor that results from choosing to predict X_t as a linear function of X_{t-1} alone; see Question 3 (e). Which of these two can be considered the "better" predictor? Justify your answer.

5) Open book question

Suppose $X_0, \dots, X_{N-1}$ are independent, identically distributed random variables with probability density function

$$f(x) = \begin{cases} \frac{1}{\lambda}\,\mathrm{e}^{-x/\lambda} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

for some (unknown) parameter $\lambda > 0$. It can be shown that $\mathrm{E}(X_t) = \lambda$ and $\mathrm{E}(X_t^2) = 2\lambda^2$.

(a) Determine the maximum likelihood estimator $\hat{\lambda}_{\mathrm{ML}}$ of λ given $X_0, \dots, X_{N-1}$.

(b) Is the estimator $\hat{\lambda}_{\mathrm{ML}}$ biased? Justify your answer.

(c) Express the Cramér-Rao lower bound for the variance of the estimator $\hat{\lambda}_{\mathrm{ML}}$ in terms of λ and N.

(d) Is there a better, unbiased estimator, i.e. an unbiased estimator with smaller variance, than $\hat{\lambda}_{\mathrm{ML}}$? Justify your answer.

(e) Is the estimator $\hat{\lambda}_{\mathrm{ML}}$ consistent? Justify your answer.

6) Open book question

Consider the seasonal model

$$X_t - c\,\mathrm{q}^{-3}X_t = \epsilon_t, \qquad t = 1,\ 2,\ \dots,$$

with c a real constant, and ϵ_t, $t \in \mathbb{N}$, a white noise process with mean 0 and variance σ_ϵ^2.

(a) Prove that the model is asymptotically stable if and only if $|c| < 1$. Assume that this condition is satisfied in the remainder of the problem.

(b) Argue that the process is asymptotically wide-sense stationary and prove that the limit of the mean value function $m_X(t) := \mathrm{E}(X_t)$ equals 0. Also prove that $\gamma(\tau) = \lim_{t\to\infty} \mathrm{Cov}(X_t, X_{t+\tau})$ is given by

$$\gamma(\tau) = \begin{cases} \frac{\sigma_\epsilon^2}{1-c^2} c^{|\tau|/3} & \text{for } |\tau| = 0,\ 3,\ 6,\ \dots, \\ 0 & \text{for other values of } \tau. \end{cases}$$

7) Open book question

Assume that $X_t = H(\mathrm{q})\epsilon_t = \sum_{k\in\mathbb{Z}} h_k \epsilon_{t-k}$, $t \in T$, is a causal, invertible, linear process and that ϵ_t, $t \in \mathbb{N}$, is a white noise process with mean 0 and variance σ_ϵ^2. In analogy to Theorem 3.6.1 in the lecture notes, formulate and prove a theorem that solves the 2-step ahead predictor problem, i.e. find the linear predictor $\hat{X}_{t|t-1} = p_2 X_{t-2} + p_3 X_{t-3} + \dots$ that minimizes the mean squared prediction error. Justify each step of the proof and specify the mean squared prediction error.

End of exam