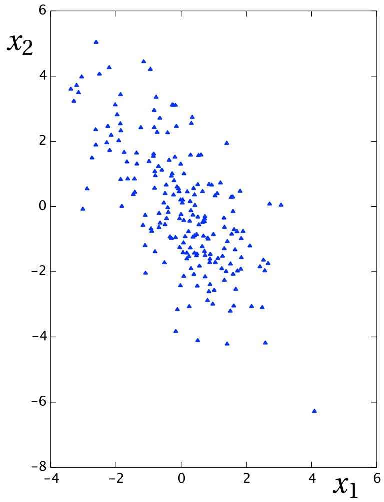
Time Series Analysis (& SI)—191571090

(Lecture notes are allowed)

Date: 08-11-2013
Place: SP1
Time: 08:45–11:45



1. (a) Consider two random variables $X = (X_1, X_2)$. Which of the following four covariance matrices

$$R_X = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}, \qquad R_X = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}, \qquad R_X = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}, \qquad R_X = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

is in accordance with the above scatter plot?

   (b) If an LTI system $\mathscr{H}$ has peak-to-peak gain 2 then the inverse system $\mathscr{H}^{-1}$ (assuming it exists) has peak-to-peak gain 1/2?

2. Consider the system

$$y_t = \frac{1}{3} y_{t-1} + u_t.$$

   (a) Is it asymptotically stable?
   (b) Determine the impulse response h
   (c) Compute $\|h\|_1$

3. Let X_t be the AR(2) process described by $(1 - a_1 \mathrm{q}^{-1} - a_2 \mathrm{q}^{-2}) X_t = \epsilon_t$ and whose covariance function $r(k)$ satisfies $r(0) = 3, r(1) = 2, r(2) = 1$.

   (a) Determine a_1, a_2
   (b) Calculate the variance σ_ϵ^2 of the white noise ϵ.
   (c) What is $r(3)$?
   (d) Suppose we have $N = 100$ samples $X_1, \ldots, X_N$ of this process and that we use it to fit an AR(2) scheme $(1 - \hat{a}_1 \mathrm{q}^{-1} - \hat{a}_2 \mathrm{q}^{-2}) X_t = \hat{\epsilon}_t$ using least-squares and where we assume that $\mathbb{E} X_t = 0$. Can you estimate $\mathrm{var}(\hat{a}_1)$ and $\mathrm{var}(\hat{a}_2)$?

4. Consider the system

$$X_t = \frac{1}{3}X_{t-2} + 2\epsilon_t + \epsilon_{t-1}$$

and assume that ϵ_t is zero mean white noise and that is has variance 1.

(a) Is the scheme invertible?

(b) Determine the one-step-ahead predictor scheme

(c) Determine the mean square prediction error $\mathbb{E}(X_t - \hat{X}_{t|t-1})^2$

5. Suppose X_t is a WSS process with covariance function $r(k)$ and consider the sample mean $\hat{m}_N = (X_1 + \cdots + X_N)/N$.

(a) Determine $\text{cov}(\hat{m}_{N+k}, \hat{m}_N)$ for the case that X_t is white noise

(b) Determine $\text{cov}(\hat{m}_{N+k}, \hat{m}_N)$ for the case that $X_t = (1 + b\,\mathrm{q}^{-1})\epsilon_t$ [THIS PROBLEM HAS BEEN DELETED. The math is too nasty.]

(c) Is $\hat{m}_t$ WSS if X_t is WSS?

6. In Example 6.5.5 the $\hat{r}_N$ is computed by inverse Fourier transformation of the periodogram ph. What is the advantage of this over direct computation of $\hat{r}_N$ via Equation (6.30)?

7. *System Identification.* Suppose we have

$$Y_t = (c_0 + c_1\,\mathrm{q}^{-1})U_t + V_t$$

and that U_t, V_t are zero mean WSS and that U and V are uncorrelated processes.

(a) We have measurements $u_0, \ldots, u_{N-1}, y_0, \ldots, y_{N-1}$. Show that for large N the least squares solution $\hat{\theta}_N := (\hat{c}_0, \hat{c}_1)$ satisfies

$$\begin{bmatrix} r_{YU}(0) \\ r_{YU}(1) \end{bmatrix} \approx \begin{bmatrix} r_U(0) & r_U(1) \\ r_U(1) & r_U(0) \end{bmatrix} \hat{\theta}_N$$

with good approximation

(b) Express $r_Y(0)$ and $r_{YU}(0), r_{YU}(1)$ in terms of r_U and r_V.

(c) Suppose in addition that V_t is white. If the power $\mathbb{E}(V_t^2)$ of the noise V_t is large then, intuitively, the estimate $\hat{\theta}_N := (\hat{c}_0, \hat{c}_1)$ is "less accurate". Argue that nonetheless $\lim_{N\to\infty} \hat{\theta}_N = (c_0, c_1)$ (using parts (a) and (b) of this exercise.)

problem:	1	2	3	4	5	6	7
points:	3+2	1+2+2	3+1+1+3	1+2+2	2+2+1	2	2+2+2

Exam grade is $1 + 9p/p_{\max}$.

1. (a) From the plot we see a negative correlation between X_1, X_2 so $\mathrm{cov}(X_1, X_2) < 0$. Hence the (1,2) en (2,1) element of R_X are < 0. The variance in X_2 is bigger than that of X_1 so $R_X(2,2) > R_X(1,1)$. So it must be $R_X = \left[\begin{smallmatrix} 2 & -2 \\ -2 & 4 \end{smallmatrix}\right]$

   (b) No: $y_t = u_t + u_{t-1}/2$ has $\|h\|_1 = 1 + 1/2$ while its inverse $u_t = y_t - y_{t-1}/2 + y_{t-2}/4 \cdots$ has 1-norm $1 + 1/2 + 1/4 + \cdots = 2$ and that is not $1/\|h\|_1$.

2. (a) Yes because the zeros of $A(z) = 1 - 1/3z^{-1}$ is $z = 1/3$ so inside unit circle

   (b) by long division of $1/(1 - \frac{1}{3}\mathrm{q}^{-1}) = 1 + 1/3\,\mathrm{q}^{-1} + 1/3^2\,\mathrm{q}^{-2} + 1/3^3\,\mathrm{q}^{-3} + \cdots$

   (c) $1 + 1/3 + 1/9 + \cdots = 1/(1 - 1/3) = 3/2$

3. (a) According to Yule-Walker we have

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -a_1 \\ -a_2 \end{bmatrix} = \begin{bmatrix} \sigma_\epsilon^2 \\ 0 \\ 0 \end{bmatrix}$$

   which implies

$$\begin{bmatrix} 0 & -4 & -8 \\ 0 & -1 & -4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -a_1 \\ -a_2 \end{bmatrix} = \begin{bmatrix} \sigma_\epsilon^2 \\ 0 \\ 0 \end{bmatrix}$$

   so $a_1 = -4a_2$ which gives $a_1 = 0.8$ and $a_2 = -0.2$

   (b) and also gives $\sigma_\epsilon^2 = 3 - 2a_1 - a_2 = 1.6$

   (c) $A(\mathrm{q})r(\tau) = 0$ for $\tau > 0$ so $r(3) = a_1 r(2) + a_2 r(1) = .4$

   (d) In lecture notes: $M = (N-n)/\sigma^2 \left[\begin{smallmatrix} r(0) & r(1) \\ r(1) & r(0) \end{smallmatrix}\right] = 98/1.6 \left[\begin{smallmatrix} 3 & 2 \\ 2 & 3 \end{smallmatrix}\right]$ hence $\mathrm{var}(\hat{a}_1) = \mathrm{var}(\hat{a}_1) = 1.6/98 \times .6 = 0.0097$

4. (a) (Invertible a la Chapter 2: yes). Invertible a la Chapter 3: also yes because the zero of $B(z) = 2 + z^{-1}$ is $z_0 = -1/2$ and $|z_0| < 1$

   (b) $B(\mathrm{q})/A(\mathrm{q}) = 2 + (1 + 2/3\,\mathrm{q}^{-1})\,\mathrm{q}^{-1}/A(\mathrm{q})$ so $B(\mathrm{q})\hat{X}_{t+1|t} = R(\mathrm{q})X_t$ becomes $(2 + \mathrm{q}^{-1})\hat{X}_{t+1|t} = (1 + 2/3\,\mathrm{q}^{-1})X_t$.

   (c) $e_t = h_0\epsilon_t = 2\epsilon_t$ so MSE is $4\sigma_\epsilon^2 = 4$.

5. The centered $\hat{m}_N - \mathbb{E}\,m_N$ equals $(\tilde{X}_1 + \cdots + \tilde{X}_N)/N$ where $\tilde{X}_t$ is the centered X_t. So without loss of generality assume $\mathbb{E}\,X_t = 0$

   (a) Suppose first that $k \geq 0$. Then $\mathrm{cov}(\hat{m}_{N+k}, \hat{m}_N) = \mathbb{E}((X_1 + \cdots + X_N)/(N+k), (X_1 + \cdots + X_N)/N) = \frac{1}{N(N+k)} N\sigma_X^2 = \frac{1}{N+k}\sigma_X^2$. For $k < 0$ (but $N + k > 0$) we have $\mathrm{cov}(\hat{m}_{N+k}, \hat{m}_N) = \mathbb{E}((X_1 + \cdots + X_{N+k})/(N+k), (X_1 + \cdots + X_{N+k})/N) = \frac{1}{N(N+k)}(N+k)\sigma_X^2 = \frac{1}{N}\sigma_X^2$.

   (b) Gratis=2 (too ugly.)

   (c) No. For white noise the $\mathrm{cov}(\hat{m}_{t+k}, \hat{m}_t)$ depends on t so not WSS

6. It is faster: fft takes $O(N \log N)$ and then ifft as well. The other commands require $O(N)$ or less so overall it takes $O(N \log(N))$ while direct computation requires $O(N^2)$.

7. (a) The lecture notes says the the least squares $\hat{\theta}_N$ satisfies

$$\frac{1}{N}F^{\mathrm{T}}Y = \frac{1}{N}F^{\mathrm{T}}F\hat{\theta}_N.$$

In our case $n = 0$ and $m = 2$. This violates the assumption $n \geq m$ made just before Equation (8.35) but we van figure out our F, Y nonetheless:

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} u_1 & u_0 \\ \vdots & \vdots \\ u_{N-1} & u_{N-2} \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}}_{\theta_N} + \begin{bmatrix} v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}$$

So $\frac{1}{N}F^{\mathrm{T}}Y = \frac{1}{N}F^{\mathrm{T}}F\hat{\theta}_N$ becomes

$$\frac{1}{N}\begin{bmatrix} u_1 & \cdots & u_{N-1} \\ u_0 & \cdots & u_{N-2} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} u_1 & \cdots & u_{N-1} \\ u_0 & \cdots & u_{N-2} \end{bmatrix} \begin{bmatrix} u_1 & u_0 \\ \vdots & \vdots \\ u_{N-1} & u_{N-2} \end{bmatrix} \hat{\theta}_N$$

That is

$$\begin{bmatrix} \frac{u_1 y_1 + \cdots u_{N-1} y_{N-1}}{N} \\ \frac{u_0 y_1 + \cdots u_{N-2} y_{N-1}}{N} \end{bmatrix} = \begin{bmatrix} \frac{u_1 u_1 + \cdots u_{N-1} u_{N-1}}{N} & \frac{u_0 u_1 + \cdots u_{N-2} u_{N-1}}{N} \\ \frac{u_0 u_1 + \cdots u_{N-2} u_{N-1}}{N} & \frac{u_1 u_1 + \cdots u_{N-1} u_{N-1}}{N} \end{bmatrix} \hat{\theta}_N$$

In expectation (for large N) this is what was asked:

$$\begin{bmatrix} r_{YU}(0) \\ r_{YU}(1) \end{bmatrix} = \begin{bmatrix} r_U(0) & r_U(1) \\ r_U(1) & r_U(0) \end{bmatrix} \hat{\theta}_N$$

(b) The lecture notes says (by the fact that U, V are uncorrelated processes) that $r_Y = r_{(c_0 + c_1 \mathrm{q}^{-1})U} + r_V$. So

$$r_Y(0) = (c_0^2 + c_1^2) r_U(0) + 2c_0 c_1 r_U(1) + r_V(0)$$

and

$$r_{YU}(0) = \mathbb{E}(c_0 U_t + c_1 U_{t-1} + V_t) U_t = c_0 r_U(0) + c_1 r_U(1)$$

and

$$r_{YU}(1) = \mathbb{E}(c_0 U_{t+1} + c_1 U_t + V_{t+1}) U_t = c_0 r_U(1) + c_1 r_U(0)$$

(c) Using the $r_{YU}(0), r_{YU}(1)$ from (b) the equation in (a) becomes

$$\begin{bmatrix} c_0 r_U(0) + c_1 r_U(1) \\ c_0 r_U(1) + c_1 r_U(0) \end{bmatrix} \approx \begin{bmatrix} r_U(0) & r_U(1) \\ r_U(1) & r_U(0) \end{bmatrix} \hat{\theta}_N$$

That is $\hat{\theta}_N = (c_1, c_2)$..