1 a) If $\mathcal{P}$ is a probability measure then we have:

$$1 = \mathcal{P}([0,1])$$
$$= \mathcal{P}\left([0,\tfrac{1}{2}] \cup [\tfrac{1}{2},1]\right)$$
$$= \mathcal{P}\left([0,\tfrac{1}{2}]\right) + \mathcal{P}\left([\tfrac{1}{2},1]\right) - \mathcal{P}\left(\{\tfrac{1}{2}\}\right)$$
$$\leqslant \frac{1}{4} + \frac{1}{4}$$

which yields a contradiction.

2) Density of W_0:

$$f_{W_0}(\omega_0) = \begin{cases} 1 & \omega \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Density of $\sqrt{X_0}$:

We have $$P(\sqrt{X_0} \leqslant a) = \begin{cases} 0 & a < 0 \\ a^2 & a \in [0,1] \\ 1 & a > 1 \end{cases}$$

Hence:

$$f_{\sqrt{X_0}}(x_0) = \begin{cases} 0 & x_0 < 0 \\ 2x_0 & x_0 \in [0,1] \\ 0 & x_0 > 1 \end{cases}$$

Density of $X_1 = \sqrt{X_0} + W_0$

$$f_{X_1}(x) = \int_{-\infty}^{\infty} f_{\sqrt{X_0}}(x-\omega)\, f_{W_0}(\omega)\, d\omega$$

$$= \int_0^1 f_{\sqrt{X_0}}(x-\omega)\, d\omega$$

$$\overset{x \in [0,1]}{=} \int_0^x 2(x-\omega)\, d\omega$$

$$= x^2$$

$$f_{X_1}(x) = \int_0^1 f_{\sqrt{X_0}}(x-\omega)\, d\omega$$

$$\overset{x \in [1,2]}{=} \int_{x-1}^1 2(x-\omega)\, d\omega$$

$$= 1-(x-1)^2$$

We find

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \in [0,1] \\ 1-(x-1)^2 & x \in [1,2] \\ 0 & x > 2 \end{cases}$$

b) The conditional density is given by:

$$f_{X_0|Y_0}(x_0|y_0) = \frac{f_{X_0,Y_0}(x_0,y_0)}{f_{Y_0}(y_0)}$$

$$= \frac{f_{X_0,V_0}(x_0, y_0-\sqrt{x_0})}{f_{Y_0}(y_0)}$$

$$= \frac{f_{X_0}(x_0)\, f_{V_0}(y_0-\sqrt{x_0})}{f_{Y_0}(y_0)}$$

Using the similarity of Y_0 and X_0 we find that

$$f_{Y_0}(y_0) = \begin{cases} 0 & y_0 < 0 \\ y_0^2 & y_0 \in [0,1] \\ 1-(y_0-1)^2 & y_0 \in [1,2] \\ 0 & y_0 > 2 \end{cases}$$

$$f_{X_0}(x_0) = \begin{cases} 0 & x_0 < 0 \\ 1 & x_0 \in [0,1] \\ 0 & x_0 > 1 \end{cases}$$

$$f_{V_0}(y-\sqrt{x_0}) = \begin{cases} 0 & y-\sqrt{x_0} < 0 \\ 1 & y-\sqrt{x_0} \in [0,1] \\ 0 & y-\sqrt{x_0} > 1 \end{cases}$$

$$E[X_0|y_0] = \int_{-\infty}^{\infty} x_0 f_{X_0|Y_0}(x_0|y_0)\, dx_0$$

$$= \int_0^1 x_0 \frac{f_{V_0}(y_0 - \sqrt{x_0})}{f_{Y_0}(y_0)}\, dx_0$$

For $y_0 \in [1,2]$ we get:

$$E[X_0|y_0] = \int_{(1-y_0)^2}^{1} \frac{x_0}{f_{Y_0}(y_0)}\, dx_0$$

$$= \frac{1}{2} \frac{1-(1-y_0)^4}{1-(1-y_0)^2}$$

$$= \frac{1}{2}\left(1+(1-y_0)^2\right)$$

For $y_0 \in [0,1]$ we get

$$E[X_0|y_0] = \int_0^{y_0^2} \frac{x_0}{y_0^2}\, dx_0$$

$$= \frac{1}{2} \frac{y_0^4}{y_0^2}$$

$$= \frac{1}{2} y_0^2$$

For $y_0 < 0$ and $y_0 > 2$ we have that $E[X_0|y_0]$ is not defined.

c) $E[X_1 | Y_0]$

$$= E[\sqrt{X_0} + W_0 | Y_0]$$

$$= E[\sqrt{X_0} | Y_0] + E[W_0]$$

$$= E[\sqrt{X_0} | Y_0] + \frac{1}{2}$$

~~Hence~~ We get:

$$E[\sqrt{X_0} | Y_0] = \int_{-\infty}^{\infty} \sqrt{x_0}\, f_{X_0|Y_0}(x_0|y_0)\, dx_0$$

$$= \int_0^1 \sqrt{x_0}\, \frac{f_{V_0}(y_0 - \sqrt{x_0})}{f_{Y_0}(y_0)}\, dx_0$$

For $y_0 \in [1,2]$:

$$E[\sqrt{X_0} | Y_0] = \int_{(1-y_0)^2}^{1} \frac{\sqrt{x_0}}{f_{Y_0}(y_0)}\, dx_0$$

$$= \cancel{\frac{2}{3}\, \frac{1-(y_0-1)^3}{y_0(2-y_0)}} = \frac{2}{3}\, \frac{1-(y_0-1)^3}{y_0(2-y_0)}$$

$$= \cancel{\frac{2}{3}\, \frac{y_0^2 - y_0 + 1}{y_0}} = \frac{2}{3}\, \frac{y_0^2 - y_0 + 1}{y_0}$$

For $y_0 \in [0,1]$

$$E[\sqrt{X_0}|y_0] = \int_0^{y_0^2} \frac{\sqrt{x_0}}{y_0^2}\, dx_0$$

$$= \frac{2}{3}\frac{y_0^3}{y_0^2}$$

$$= \frac{2}{3} y_0$$

$$E[X_1|y_0] = \begin{cases} \frac{2}{3} y_0 + \frac{1}{2} & 0 \leq y_0 \leq 1 \\ \frac{2}{3}\frac{y_0^2 - y_0 + 1}{y_0} + \frac{1}{2} & 1 \leq y_0 \leq 2 \end{cases}$$

3) We have $E[X_0|y_0] = y_0 = \hat{X}_{0|0}$

and $E(X_0 - E[X_0|y_0])^2 = P_{0|0} = 1$

We get:

$$E[X_1|y_0] = y_0^2$$

Linearizing we get:

$$X_1 \approx y_0^2 + 2y_0(X_0 - \hat{X}_{0|0}) + W_0$$

$$A_0 = 2y_0$$

$$P_{1|0} = (2y_0) P_{0|0} (2y_0) + 1$$
$$= 4y_0^2 + 1$$

$$K_1 = P_{1|0} \left(P_{1|0} + 1 \right)^{-1}$$
$$= \frac{4y_0^2+1}{4y_0^2+2}$$

$$P_{1|1} = P_{1|0} - K_1 P_{1|0} = 4y_0^2 + 1 - \frac{4y_0^2+1}{4y_0^2+2}(4y_0^2+1)$$
$$= \frac{4y_0^2+1}{4y_0^2+2}$$

$$E[x_1 \mid y_0, y_1]$$
$$= y_0^2 + \frac{4y_0^2+1}{4y_0^2+2} \left[y_1 - \hat{x}_{1|0} \right]$$
$$= y_0^2 + \frac{4y_0^2+1}{4y_0^2+2} \left[y_1 - y_0^2 \right]$$
$$= \frac{1}{4y_0^2+2} y_0 + \frac{4y_0^2+1}{4y_0^2+2} y_1$$

4 a) We know $p(x_k \mid x_{k-1}^i, y_k)$ is Gaussian.

We need to compute its expectation and variance

*) $E[x_k \mid x_{k-1}^i, y_k]$

$= E[x_{k-1}^i + w_{k-1} \mid x_{k-1}^i, y_k - x_{k-1}^i]$

$= x_{k-1}^i + E[w_{k-1} \mid x_{k-1}^i, v_k + w_{k-1}]$

$= x_{k-1}^i + E[w_{k-1} \mid v_k + w_{k-1}]$

$= x_{k-1}^i + \frac{1}{2}[v_k + w_{k-1}]$

$= \frac{1}{2} x_{k-1}^i + \frac{1}{2} y_k$

*) $\mathrm{Var}(x_k - \frac{1}{2} x_{k-1}^i - \frac{1}{2} y_k)$

$= \mathrm{Var}(\frac{1}{2} w_{k-1} + \frac{1}{2} v_{k-1}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

So x_k^i should be drawn from a Gaussian distribution. with mean $\frac{1}{2} x_{k-1}^i + \frac{1}{2} y_k$ and variance $\frac{1}{2}$

4 b)

$$X_k^i = \frac{1}{2} X_{k-1}^i + \frac{1}{2} Y_k + S_k$$

with S_k Gaussian with mean 0
and variance $\frac{1}{2}$

$$X_k^i - X_k = \frac{1}{2} X_{k-1}^i + \frac{1}{2} Y_k + S_k - X_k$$

$$= \frac{1}{2} (X_{k-1}^i - X_{k-1}) + \frac{1}{2} V_k + \frac{1}{2} W_{k-1} + S_k$$

$$\mathrm{Var}(X_k^i - X_k) = \frac{1}{4} \mathrm{Var}(X_{k-1}^i - X_{k-1}) + \frac{1}{4} + \frac{1}{4} + \frac{1}{2}$$

$$= \frac{1}{4} \mathrm{Var}(X_{k-1}^i - X_{k-1}) + 1$$

4c) From 4 b) we find that ~~$\mathrm{Var}(X_k^i - X_k)$~~ $\mathrm{Var}(X_k^i - X_k)$ is bounded in k.

This implies that there always ~~are~~ must be a reasonable fraction of particles close to X_k. Hence resampling will not be needed.