UNIVERSITEIT TWENTE

Faculteit Elektrotechniek, Wiskunde en Informatica

Test exam 2, Random Signals and Filtering (201200135)

The solutions of the exercises need to be clearly formulated and written in a well-structured manner. Moreover, you always need to present a derivation or arguments to support your answer.

You can use one single-sided A4 page of handwritten notes with your exam.

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1. Consider

$$f(x,y) = \begin{cases} ax + \frac{1}{8}y + \frac{1}{4} & |x| < 1, |y| < 1 \\ 0 & \text{otherwise} \end{cases}$$

   a) Determine all values of $a \in \mathbb{R}$ for which f can be a valid density function.

   Choose $a = 0$. Let X and Y be real-valued stochastic variables with joint density function f.

   b) Determine whether X and Y are independent.

   c) Determine the expectation and variance of X and Y.

2. Consider

$$f(x,y) = \begin{cases} \frac{3}{8}(x^2 + y^2) & |x| < 1, |y| < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Let X and Y be real-valued stochastic variables with joint density function f. Define $Z = E[X|Y]$.

   a) Determine the expectation of Z.

   b) Determine the variance of Z.

3. Consider the following linear system:

$$\begin{aligned} X_{k+1} &= X_k + W_k \\ Y_k &= X_k + 2V_k \end{aligned}$$

   where X_0, V_k and W_k are mutually independent and all have a uniform distribution on the interval $[-1, 1]$. Moreover, the noise sequences $\{W_k\}$ and $\{V_k\}$ are assumed to be white.

   a) Determine $Z = E_{\text{lin}}[X_4|Y_0, Y_1, Y_2]$

   b) Determine the expectation of Z

   c) Determine the variance of Z.

4. Consider the following linear system:

$$\begin{aligned} X_{k+1} &= X_k + W_k \\ Y_k &= X_k + V_k \end{aligned}$$

where X_0, V_k and W_k are mutually independent and all have a Gaussian distribution with mean zero and variance 1. Moreover, the noise sequences $\{W_k\}$ and $\{V_k\}$ are assumed to be white.

We are applying a particle filter without resampling where we recursively update our particles according to:

$$\pi(x_k \mid x_{k-1}^i, y_k) = p(x_k \mid x_{k-1}^i)$$

At time k we will have particles $X_k^1, X_k^2, \ldots, X_k^N$.

a) Show that X_k^i has variance $k+1$.

b) Show that $E[X_k \mid Y_0, \ldots, Y_k]$ has a variance less than 1. Hint: first investigate $E[X_k \mid Y_k]$.

c) Show that

$$\sum_{i=1}^{N} q_i X_k^i$$

with $\sum_{i=1}^{N} q_i = 1$ has a variance of at least $\frac{k}{N}$.

d) Argue why a particle filter without resampling does not work well in this case.