Random Signals and Filtering (201200135)

Final Exam (with 4 questions) Full Marks: 30
Thursday 14/04/2016, 13:45 – 16:45 Instructor: P. K. Mandal

Formulate your answers clearly and present them in a well-structured manner.

1. Prove that the intersection of two σ-fields is again a σ-field. [4]

2. Recall the (semi-)measure theoretic definition of conditional expectation:
For an integrable r.v. X, i.e., $E(|X|) < \infty$, $E(X|Y)$ is defined to be the function $h(Y)$ where $h : \mathbb{R} \to \mathbb{R}$ is (Borel) measurable and satisfies

$$E\big[h(Y)\,g(Y)\big] = E\big[X\,g(Y)\big] \qquad \text{for all bounded mesurable function } g : \mathbb{R} \to \mathbb{R}.$$

To determine the function $h(\cdot)$ in the definition above one often resorts to the non-measure theoretic definition of conditional probability distribution.

(a). Consider square integrable r.v.s U, V and W, where U is independent of V and W. Show from the (measure theoretic) definition that $E(U\,V|\,W) = E(U)\,E(V|\,W)$. [3]

(b.) Consider the following nonlinear system: for $k \geq 0$,

$$\begin{aligned} X_{k+1} &= X_k\,\sqrt{W_k} \\ Y_k &= \sqrt{X_k}\,V_k, \end{aligned}$$

where the initial state X_0, and for $k \geq 0$, the noises W_k, V_k are all Uniform$(0,1)$. Furthermore, X_0, $\{W_k\}$, and $\{V_k\}$ are mutually independent and the noise sequences $\{W_k\}$ and $\{V_k\}$ are white.

(i) Determine $E(X_0\,|\,Y_0)$. [3]

(ii) Determine $E(X_1\,|\,Y_0)$. [2]

3.(a.) What is meant by linear innovations corresponding to a sequence of measurements $Y_0, Y_1, \ldots, Y_n$? Show that linear innovations are uncorrelated. [3]

(b.) Obtain the first two innovations for the following nonlinear system: for $k \geq 0$, [3]

$$\begin{aligned} X_{k+1} &= X_k^2 + W_k \\ Y_k &= X_k + V_k, \end{aligned}$$

where the initial state X_0, and for $k \geq 0$, the noises W_k, V_k are all $N(0,1)$. Furthermore, X_0, $\{W_k\}$, and $\{V_k\}$ are mutually independent and the noise sequences $\{W_k\}$ and $\{V_k\}$ are white.

4. Recall that in a particle filter algorithm at time k, x_k^i is drawn from a proposal/importance density $\pi(x_k; x_{k-1}^i, y_k)$ and subsequently the weights are updated. The weight update step involves the calculation of the unnormalized weights:

$$\tilde{w}_k^i = w_{k-1}^i \frac{p(y_k|x_k^i)\, p(x_k^i|x_{k-1}^i)}{\pi(x_k^i; x_{k-1}^i, y_k)}.$$

(a.) Show that this equation becomes $\tilde{w}_k^i = w_{k-1}^i\, p(y_k|x_{k-1}^i)$ when one uses the "optimal importance density", i.e., $\pi(x_k; x_{k-1}, y_k) = p(x_k|x_{k-1}, y_k)$. [2]

In general, it is not easy to determine the optimal importance density $p(x_k|x_{k-1}, y_k)$. Neither is it easy to draw samples from it. However, it is possible to do so when the measurement equation is linear-Gaussian.

Consider again the (real-valued) nonlinear system in question 3(b).

(b.) Argue that given $X_{k-1} = x_{k-1}$, the (conditional) joint distribution of X_k and Y_k is Gaussian and determine it completely. [3]

(c.) Determine $p(x_k|y_k, x_{k-1})$ using the following fact. [2]

If X and Y are random vectors distributed jointly as Gaussian, then the conditional distribution of X given $Y = y$ is Gaussian with mean and covariance (matrix), given respectively by

$$\begin{aligned} \mu(y) &= E(X) + \mathrm{Cov}(X,Y)\,\mathrm{Cov}(Y)^{-1}\,(y - E(Y)) \\ \Sigma(y) &= \mathrm{Cov}(X) - \mathrm{Cov}(X,Y)\,\mathrm{Cov}(Y)^{-1}\,\mathrm{Cov}(Y,X). \end{aligned}$$

(d.) Give the pseudo code for a generic iteration step of the particle filter, i.e., how to obtain $(x_k^i, w_k^i)_{i=1}^N$ from $(x_{k-1}^i, w_{k-1}^i)_{i=1}^N$ and the measurement y_k. [5]

Assume that you have access to a command `normpdf(x,m,s)` allowing you to evaluate the normal pdf (with mean `m` and variance `s`^{2}) at point `x` and the command `randn(m,s)` to generate a sample from it.