Random Signals and Filtering (201200135)

Final Exam (with 4 questions) | Full Marks: 25
Thursday 20/04/2017, 13:45 – 16:45 | Instructor: P. K. Mandal

Formulate your answers clearly and present them in a well-structured manner.

1.(a). Consider $\Omega = [0, 1]$ and let $\mathcal{P}$ be defined on subsets of Ω such that

$$\mathcal{P}([a, b]) = (b - a)^2, \qquad \forall\, 0 \leq a < b \leq 1.$$

Verify whether $\mathcal{P}$ satisfies all the axioms of a probability measure. [3]

(b.) Suppose $X \sim \text{Unif}(0, \pi)$ is independent of $V \sim \text{Unif}(-\frac{1}{2}, \frac{1}{2})$ and $Y = \cos(X) + V$. Find the probability density function of Y. [4]

[Hint: Express the joint density $f_{X,Y}(x, y)$ as compactly as possible using the conditional pdf $f_{Y|X=x}(y)$. While marginalizing, consider the three cases $y < -\frac{1}{2}$, $y > \frac{1}{2}$ and the rest, separately and also, use the graph of $\cos(\cdot)$ function to decide the proper ranges of integration.]

2. Recall the definition of the (orthogonal) projection:

The projection, $\Pi_{\mathcal{V}} X$, of a square integrable r.v. X on to the closed linear subspace $\mathcal{V} \subset L^2(\Omega, \mathcal{F}, P)$ is the r.v. $\hat{X} \in \mathcal{V}$ satisfying

$$E\left[(X - \hat{X})^2\right] \leq E\left[(X - Z)^2\right] \quad \forall\, Z \in \mathcal{V}. \qquad (\dagger)$$

In the following, you are going to prove the *Projection Theorem*:
Let $X^* \in \mathcal{V}$. Then $X^* = \Pi_{\mathcal{V}} X$ iff (if and only if)

$$E\left[(X - X^*)\, Z\right] = 0 \quad \forall\, Z \in \mathcal{V}. \qquad (\star)$$

(a). Show that if $X^* \in \mathcal{V}$ and $(\star)$ holds then $X^* = \Pi_{\mathcal{V}} X$. [2]

(b.) For the "only if" part, suppose $X^* = \Pi_{\mathcal{V}} X$.
First, show by considering r.v.s of the form $X^* + c\, Z$ that [2]

$$c^2\, E[Z^2] - 2\, c\, E[(X - X^*)\, Z] \geq 0 \qquad \forall\, c \in \mathbb{R},\ Z \in \mathcal{V}.$$

Next, complete the square w.r.t. c to rewrite the inequality in the form: [1]

$$\left(c \cdot \text{expression}_1 - \text{expression}_2\right)^2 - \text{expression}_3 \geq 0.$$

Finally, argue that $(\star)$ must hold. [1]

3.(a.) Show that $p(x|y,z) = \dfrac{p(x|z)\,p(y|x,z)}{p(y|z)}$. [2]

In above we have used the shorthand notation. For example, $p(y|x,z)$ stands for the conditional probability density function of the r.v. Y given $X = x$ and $Z = z$, i.e., $f_{Y|\{X=x,Z=z\}}(y)$.

(b.) Consider a standard 1st order (partially observed) state space model of the form

$$\text{(state)} \quad x_k = f_{k-1}(x_{k-1}, u_{k-1}), \qquad \text{(measurement)} \quad y_k = h_k(x_k, v_k),$$

where the noises $\{u_k\}$ and $\{v_k\}$ are independent.

In the derivation of the *weight update step* of the standard particle filter algorithm, using the importance sampling technique, one comes across the following equation:

$$w_k^i = w_{k-1}^i \frac{p(x_{0:k}^i|y_{0:k})}{p(x_{0:k}^i|y_{0:k-1})}$$

where $\{x_{0:k-1}^i, w_{k-1}^i\}_{i=1}^N$ is the particle representation of $p(x_{0:k-1}|y_{0:k-1})$.

Show that the weight update equation reduces to $w_k^i = w_{k-1}^i \dfrac{p(y_k|x_k^i)}{p(y_k|y_{0:k-1})}$. [3]

[Hint: Note that $p(x_{0:k}^i|y_{0:k}) = p(x_{0:k}^i|y_k, y_{0:k-1})$.]

4. Consider the following nonlinear system: for $k \geq 0$,

$$\begin{aligned} X_{k+1} &= \frac{1}{2}X_k + W_k \\ Y_k &= \cos(X_k) + V_k, \end{aligned}$$

where the initial state $X_0 \sim \text{Unif}(0,\pi)$, $W_k \sim \text{Unif}(0,\pi/2)$, and $V_k \sim \text{Unif}(-\frac{1}{2},\frac{1}{2})$. Furthermore, X_0, $\{W_k\}$, and $\{V_k\}$ are mutually independent and the noise sequences $\{W_k\}$ and $\{V_k\}$ are white.

(a.) A person is thinking about applying the extended Kalman Filter to the system, because: *the state equation is already linear; only nonlinearity is in the measurement equation, which can be easily linearized.*

Reflect on the provided argument. If you agree, complement the argument with the exact form of the linearized equation. If you do not agree, then justify. [2]

(b.) Give the pseudo-code for a generic iteration step of the particle filter, i.e., how to obtain $(x_k^i, w_k^i)_{i=1}^N$ from $(x_{k-1}^i, w_{k-1}^i)_{i=1}^N$ and the measurement y_k. [5]

Assume that you have access to a command `rand(a,b)` to generate a sample from Unif(a,b) distribution. Everything else in your code should be self explanatory, i.e., in terms of known function such as $\cos(\cdot)$ or known structures like `if` and `for` loop.