Random Signals and Filtering (201200135)

Final Exam (with 4 questions) — Full Marks: 35
Thursday 27/07/2017, 13:45 – 16:45 — Instructor: Dr. P. K. Mandal

Formulate your answers clearly and present them in a well-structured manner.

1. Let $\mathcal{F}$ be a σ-field on Ω, X be a random variable defined on $(\Omega, \mathcal{F})$, and $\mathfrak{B}(\mathbb{R})$ be the Borel σ-field on $\mathbb{R}$. Consider the following collection of inverse images under X:

$$\mathbb{A} = \{X^{-1}(B) : B \in \mathfrak{B}(\mathbb{R})\}.$$

Show that $\mathbb{A}$ is a σ-field. [4]

[Hint: Recall that $f^{-1}\big(A^c\big) = \big(f^{-1}(A)\big)^c$ and $f^{-1}\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$.]

2. Let X and Y be two random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Recall that $E_{\text{aff}}(X \,|\, Y)$ can be thought of as the (orthogonal) projection of X onto the space of linear functions of Y, of the form $a + bY$. You will now derive the formula for $E_{\text{aff}}(X \,|\, Y)$.

   (a) Use the *Projection Theorem* to show that $E_{\text{aff}}(X \,|\, Y) = a^* Y + b^*$, where a^* and b^* satisfy the normal equations: [3]

$$\begin{aligned} E[X] &= a^* E[Y] + b^* \\ E[XY] &= a^* E[Y^2] + b^* E[Y]. \end{aligned}$$

   (b) Show that $E_{\text{aff}}(X \,|\, Y) = E(X) + \text{Cov}(X,Y)\,\text{Var}(Y)^{-1}\,(Y - E(Y)).$ [1]

3. Consider the following nonlinear system: for $k \geq 0$,

$$X_{k+1} = \sqrt{X_k}\, W_k \quad \text{and} \quad Y_k = X_k \sqrt{V_k},$$

where the initial state X_0 and the noises W_k, V_k $(k \geq 0)$ are all Uniform$(0,1)$. Furthermore, X_0 and the sequences $\{W_k\}$ and $\{V_k\}$ are mutually independent.

   (a) Suppose $0 < x < 1$. Show that, for any $k \geq 0$, the conditional probability density function *(pdf)* of Y_k given $\{X_k = x\}$ is $f_{Y_k|X_k=x}(y) = \dfrac{2y}{x^2}$, $0 < y < x$. [2]

   (b) Show that the joint *pdf* of (X_0, Y_0) and the *pdf* of Y_0 are given by, respectively, [3]

$$f_{X_0,Y_0}(x_0, y_0) = \frac{2\,y_0}{x_0^2}, \; 0 < y_0 < x_0 < 1, \quad \text{and} \quad f_{Y_0}(y_0) = 2(1 - y_0), \; 0 < y_0 < 1.$$

   (c) Determine $E(X_0 \,|\, Y_0)$. [3]

   (d) Calculate $E_{\text{aff}}(X_0 \,|\, Y_0)$ and relate it to the answer in part (c). [4]

4. Consider, again, the nonlinear system as described in question 3. In other words,

$$\begin{aligned} X_{k+1} &= \sqrt{X_k}\, W_k \\ Y_k &= X_k \sqrt{V_k}, \end{aligned}$$

where the initial state X_0 and the noises W_k, V_k $(k \geq 0)$ are all Uniform$(0,1)$. Furthermore, X_0 and the sequences $\{W_k\}$ and $\{V_k\}$ are mutually independent.

(a) Suppose we want to implement a particle filter (PF) to this system with the importance density $\pi(x_k; x_{k-1}, y_k) = p(x_k|x_{k-1})$. Describe, given the weighted particle representation $\{(x_{k-1}^i, w_{k-1}^i),\ i = 1, 2, \ldots, N\}$ of the posterior distribution at time $(k-1)$, and the new measurement y_k at time k, how you can obtain the particle representation, $\{(x_k^i, w_k^i),\ i = 1, 2, \ldots, N\}$, of the posterior at time k. Do this in the form of an algorithm/pseudo-code. [5]

Assume that you have access to a command `rand(a,b)` to generate a sample from Unif(a, b) distribution. Everything else in your code should be self explanatory, i.e., in terms of standard arithmetic operations and known structures like `if` and `for` loop.

(b) How can you extract the posterior mean $E(X_k \mid Y_{0:k})$ and the variance $\mathrm{Var}(X_k \mid Y_{0:k})$ from the particle representation? [2]

(c) Often a resampling step is performed in a PF algorithm. Discuss briefly the problem that the resampling step tries to solve. [2]

(d) Often one talks about the *optimal* importance/proposal density in a PF. What is it? In what sense is it optimal? [2]

(e) Could we have used (extended) Kalman filter (EKF) for this system to extract the posterior information about x_k? If yes, how would you proceed? If not, what would you have needed to be able to apply EKF? [4]