# Random Signals and Filtering (201200135)

Faculty of EEMCS, University of Twente

Final Exam (with 4 questions) — Full Marks: 30

Thursday, 18/04/2019, 13:45 – 16:45 — Instructor: P. K. Mandal

This is a closed book exam. Formulate your answers clearly, with proper motivation. Present your answers in a well-structured manner.

1. Suppose $\Omega = [0, \infty)$. Consider a function $\mathcal{P}$, defined on (Borel) subsets of $\Omega$, satisfying:

$$\mathcal{P}\Big( [n,\, n+1) \Big) = \frac{1}{2^{n+1}} \quad \text{and} \quad \mathcal{P}\Big( [n,\, n+1] \Big) = \frac{3}{2^{n+2}}, \quad \text{for } \ n = 0, 1, 2, \ldots$$

   a) Analyze if $\mathcal{P}$ can satisfy all the axioms of a probability measure. [3]

   b) Suppose $\mathcal{P}$ does satisfy all the axioms, and let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers. Calculate $\mathcal{P}(\ \Omega \setminus \mathbb{N}\ )$. [2]

2. a) Suppose $X$ is a square integrable r.v., i.e., $E(X^2) < \infty$. Recall that $E(X \mid Y)$ can be thought of as the projection of $X$ onto $\mathcal{V}$, the space of square integrable (Borel) functions of $Y$.

   Consider square integrable r.v.s $U$, $V$ and $W$, where $U$ is independent of $V$ and $W$. Use the *Projection Theorem* to show that $E(U\,V \mid W) = E(U)\,E(V \mid W)$. [3]

   b) Consider the following nonlinear system: for $k \geq 0$,

$$X_{k+1} \;=\; X_k \sqrt{W_k} \qquad \text{and} \qquad Y_k \;=\; \sqrt{X_k}\, V_k,$$

   where the initial state $X_0$, and the noises $W_k$, $V_k$, for $k \geq 0$, are all Uniform$(0,1)$. Furthermore, $X_0$, $\{W_k\}$, and $\{V_k\}$ are mutually independent and the noise sequences $\{W_k\}$ and $\{V_k\}$ are white.

   Note that there is an observation $Y_0$ at $t = 0$! In the following, you will obtain the first "filtered" estimate $\hat{X}_0$ and the next "predicted" estimate $\hat{X}_{1|0}$.

   (*i.*) Show that the pdf of $Y_0$ and the posterior pdf of $X_0$ (conditional on $Y_0 = y_o$) are given by

$$p_{Y_0}(y_0) \;=\; 2(1 - y_0), \ \text{ for } y_0 \in (0,1) \qquad \text{(and 0, otherwise.)}$$

$$p(x_0|y_0) \;=\; \begin{cases} \text{undefined}, & \text{if } \ y_0 \notin (0,1) \\ \frac{1}{2\sqrt{x_0}\,(1-y_0)}, & \text{for } \ y_0^2 \leq x_0 \leq 1, \ \text{ provided } \ y_0 \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

   Determine, also, $\hat{X}_0 := E(X_0 \mid Y_0)$. [2+2+2]

   (*ii.*) Use part (a), to determine $\hat{X}_{1|0} := E(X_1 \mid Y_0)$. [2]

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✓ 3. a) What is meant by linear innovations corresponding to a sequence of measurements $Y_0, Y_1, \ldots, Y_n$? Show that any two linear innovations are uncorrelated. [3]

✓ b) Consider the real-valued nonlinear system in Question 4, with $\sigma_0 = \sigma_s = \sigma_m = 1$. Obtain the first two linear innovations for the sequence of measurements generated by the system. [3]

[ Hint: Use different properties of covariance and the following facts.
For $Z \sim \mathcal{N}(0,1)$, $E(Z^4) = 3$ and $E(Z^6) = 15$.
Also, $E_{\text{aff}}(X \mid Y) = E(X) + \operatorname{Cov}(X,Y)\operatorname{Cov}(Y)^{-1}(Y - E(Y))$. ]

4. Consider the following (real-valued) nonlinear system: for $k \geq 0$,

$$\begin{aligned} X_{k+1} &= X_k^2 + \xi_k \\ Y_k &= X_k^2 + V_k, \end{aligned}$$

where the initial state $X_0 \sim \mathcal{N}(0, \sigma_0^2)$, and for $k \geq 0$, the state noises $\xi_k \sim \mathcal{N}(0, \sigma_s^2)$ and the measurement noises $V_k \sim \mathcal{N}(0, \sigma_m^2)$. Furthermore, $X_0$, $\{\xi_k\}$, and $\{V_k\}$ are mutually independent and the noise sequences $\{\xi_k\}$ and $\{V_k\}$ are white.
Note that there is an observation $Y_0$ at $t = 0$!

a) Suppose we would like to implement a particle filter (PF) to the system with the importance density $\pi(x_k; x_{k-1}, y_k) = p(x_k|x_{k-1})$. Give the pseudo code for a generic iteration step of the particle filter. [4]

More precisely, describe how you will use $\{(x_{k-1}^i, w_{k-1}^i),\ i = 1, 2, \ldots, N\}$, the weighted particle representation of the posterior at time $(k-1)$, and the measurement $y_k$ at current time $k$, to obtain the particle representation of the current posterior: $\{(x_k^i, w_k^i),\ i = 1, 2, \ldots, N\}$.

The code should be self explanatory, i.e., in terms of known/standard functions or known structures like `if` and `for` loop. You may assume that you have access to the following commands.

`NormPDF(x,m,s)` that evaluates the $\mathcal{N}(\mathtt{m}, \mathtt{s}^2)$-density at the point `x`

`RandNorm(m,s)` that generates a sample from a $\mathcal{N}(\mathtt{m}, \mathtt{s}^2)$-r.v.

`RandPMF(x_vec, p_vec, n)` that produces a random sample of size "`n`" from the discrete distribution with values "`x_vec`" and corresponding probabilities "`p_vec`".

b) How will you extract the posterior mean $E(X_k \mid Y_{0:k} = y_{0:k})$ and the posterior variance $\operatorname{Var}(X_k \mid Y_{0:k} = y_{0:k})$ from the particle representation? [2]

c) Often, one considers other importance/proposal densities than the standard one used in above, i.e., the state-transition density $p(x_k|x_{k-1})$. What are the advantages and disadvantages of the standard proposal? [2]