# Online Exam: Random Signals and Filtering (201200135)
# Thursday, 16/04/2020, 13:45 – 16:45

- You are allowed to use the two reference books/notes (by Hajek and Stoorvogel), the notes provided by me as well as those you took during the classes.
- All answers must be motivated; arguments and proofs must be complete. You may use any result you like without proof, unless asked to prove explicitly. You should aways provide references to te used results. References must be to the textbook(s) and/or lecture slides.
- At the end of this written exam, there will be a supplementary oral exam for a group of randomly selected students. The oral will last maximum 15 minutes per student. The oral will be recorded on video and will only be used for quality control (by, e.g., the examination board, accreditation committee and the likes).
- You have the right to refuse the oral exam being recorded. In that case, however, you may not participate in this online exam. You have to wait till the situation (around Corona-virus) normalizes and another on-campus exam can be scheduled.

The written test is worth 30 points.

Your grade on the written exam will be: $1 + \frac{\text{obtained\_points}}{30} \times 9.$

Please read carefully: *By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test. If significant irregularities are detected, the examiner/examination board may declare the test result of an individual or those for all participants invalid.*

In order for the test to be graded, the following text must be copied (handwritten and signed) on the first page of your solution, along with a photocopy of your student-id:

> **"I have made this test to the best of my own ability, without seeking or accepting help of any source not explicitly allowed by the conditions of the test. Neither have I provided help to anybody else.**
> **Also, I give my consent to the oral exam being recorded on video for the purpose of quality control."**
> **[Name, Student no., Location, Date, Signature]**

Questions start on the next page.

**[ There are 6 questions (with 11 parts in total) Full Marks : 30 ]**

1. Consider $\Omega = [0, 1]$ and the Borel $\sigma$-algebra, $\mathcal{B}$, defined on $\Omega$. Let $\mathcal{P}$ be defined on $\mathcal{B}$ such that on the closed intervals it is given by

$$\mathcal{P}([a, b]) = (b - a)^2, \qquad \forall\, 0 \le a \le b \le 1.$$

Analyze if $\mathcal{P}$ satisfies all the axioms of a probability measure. [3]

2. Suppose $X$ is a square integrable r.v., i.e., $E(X^2) < \infty$. Recall that $E(X \mid Y)$ can be thought of as the projection of $X$ onto $\mathcal{V}$, the space of square integrable (Borel) functions of $Y$.

Use characterizing properties of projection to show that [4]

$$E(X\, h(Y) | \, Y) = h(Y)\, E(X | \, Y), \qquad \text{for all Borel square integrable functions } h(\cdot).$$

3. Let $\begin{pmatrix} X \\ Y \end{pmatrix}$ be a 2-dimensional Gaussian vector with mean zero (vector) and variance-covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, where $|\rho| < 1$. In the following you will obtain <u>*the best*</u> and <u>*the best affine*</u> estimator of $X^2$ based on $Y$.

   a) Show that the best estimator of $X^2$ based on $Y$, i.e., $E(X^2|Y) = 1 - \rho^2 + \rho^2\, Y^2$. [2]

   [Hint: Recall the definition of conditional variance: $\text{Var}(U|V) = E(U^2|V) - \big(E(U|V)\big)^2$ ]

   b) Show that $E(X^2 Y) = 0$. [2]

   [Hint: Law of iterated expectation/tower law, and other properties of conditional expectation, together with the fact that $Y$ is a symmetric random variable can be useful.]

   c) Obtain the best *affine* estimator of $X^2$ based on $Y$, i.e., $E_{\text{aff}}(X^2|Y)$. [3]

4. a) Show, from the definition of conditional density, that $\quad p(x|y, z) = \dfrac{p(x|z)\, p(y|x, z)}{p(y|z)}$. [2]

   In above, we have used the shorthand notation. For example, $p(y|x, z)$ stands for the conditional probability density function of the r.v. $Y$ given $X = x$ and $Z = z$, i.e., $f_{Y|\{X=x,Z=z\}}(y)$.

   b) Consider a standard 1st order (partially observed) state space model of the form

$$\text{(state)} \quad x_k = f_{k-1}(x_{k-1}, u_{k-1}), \qquad \text{(measurement)} \quad y_k = h_k(x_k, v_k),$$

   where the noises $\{u_k\}$ and $\{v_k\}$ are independent.

   In a PF algorithm, at time $k$, $x_k^i$'s are drawn from a proposal/importance density $\pi(x_k; x_{k-1}^i, y_k)$ and subsequently, the weights are updated. The weight update step involves the calculation of the unnormalized weights:

$$\tilde{w}_k^i = w_{k-1}^i \, \frac{p(y_k|x_k^i)\, p(x_k^i|x_{k-1}^i)}{\pi(x_k^i; x_{k-1}^i, y_k)}.$$

   Show that this equation becomes $\quad \tilde{w}_k^i = w_{k-1}^i\, p(y_k|x_{k-1}^i)$ when one uses the "optimal importance density", i.e., $\pi(x_k; x_{k-1}, y_k) = p(x_k|x_{k-1}, y_k)$. [3]

5. Consider the following nonlinear system: for $k \geq 0$,

$$
\begin{aligned}
X_{k+1} &= \frac{1}{2} X_k + W_k \\
Y_k &= \cos(X_k) + V_k,
\end{aligned}
$$

where the initial state $X_0 \sim \text{Unif}(0, \pi)$, $W_k \sim \text{Unif}(0, \pi/2)$, and $V_k \sim \text{Unif}(-\frac{1}{2}, \frac{1}{2})$. Furthermore, $X_0$, $\{W_k\}$, and $\{V_k\}$ are mutually independent and the noise sequences $\{W_k\}$ and $\{V_k\}$ are white.

a) A person is thinking about applying the extended Kalman Filter to the system, because: *the state equation is already linear; only nonlinearity is in the measurement equation, which can be easily linearized.*

   Reflect on the provided argument. If you agree, complement the argument with the exact form of the linearized equation. If you do not agree, then justify. [2]

b) Suppose we would like to implement a standard particle filter (PF) to the system, i.e., with the importance density $\pi(x_k; x_{k-1}, y_k) = p(x_k | x_{k-1})$. Give the pseudo code for a generic iteration step of the particle filter. [4]

   More precisely, describe how you will use $\{(x_{k-1}^i, w_{k-1}^i),\ i = 1, 2, \ldots, N\}$, the weighted particle representation of the posterior at time $(k-1)$, and the measurement $y_k$ at current time $k$, to obtain the particle representation of the current posterior: $\{(x_k^i, w_k^i),\ i = 1, 2, \ldots, N\}$.

   The code should be self explanatory, i.e., in terms of known/standard functions or known structures like `if` and `for` loop. You may assume that you have access to the following commands. `rand(a,b)` that generates a sample from a $\text{Unif}(a, b)$ distribution.

   `RandPMF(x_vec, p_vec, n)` that produces a random sample of size "`n`" from the discrete distribution with values "`x_vec`" and corresponding probabilities "`p_vec`".

c) If you are interested in the posterior probability $P(\frac{\pi}{4} \leq X_k \leq \frac{3\pi}{4} \,|\, Y_{0:k} = y_{0:k})$, how can you obtain this using the running PF? Justify your answer. [2]

6. In the Kalman Filter algorithm, two steps are used, iteratively, to arrive at the new estimate from the previous estimate, namely the prediction step and the update step. For every step, discuss briefly, whether the state estimate, produced by that step, reduces or increases the uncertainty in the state estimate (as compared to the estimate the step started with). [3]