**Mastermath and LNMB Course: Discrete Optimization**

# Exam 6 January 2014

**Utrecht University, Educatorium, 13:30–16:30**

The examination lasts 3 hours. Grading will be done before January 20, 2014. Students interested in checking their results can make an appointment by e-mail (g.schaefer@cwi.nl).

The examination consists of six problems. The maximum number of points to be gained on the different parts are as indicated below.

| 1(a)–(j) | 2 | 3 | 4 | 5 | 6(a) | 6(b) | $\Sigma$ |
|---|---|---|---|---|---|---|---|
| 20 (2 each) | 10 | 15 | 10 | 10 | 15 | 10 | **90** |

The grade for the exam is obtained by dividing the total number of points by **9**. This implies that **49.5** points are needed to pass.

During the exam, only the Lecture Notes and the assignments (incl. solutions) are allowed to be on your desk and all electronic equipment must be switched off. **No additional material is permitted**. The Lecture Notes may contain annotations that you made during the lectures. You may use the assignments with the solutions that were provided (the solutions that you handed in are not allowed).

Please be short, clear and precise in your answers. If you use results from the Lecture Notes, please provide the respective references.

Please hand in your answers **together** with the exam sheet.

Wishing you a Happy New Year 2014 and Good Luck!

**Problem 1** (20 points (2 points each)). State for each of the claims below whether it is **true** of **false**. Note: You need not justify or prove your answers here.

(a) $\sqrt{n} = \Omega(n)$.

(b) Given a spanning tree $T$ of an undirected graph $G = (V,E)$, let $\deg_T(u)$ refer to the number of edges in $T$ that are incident to $u \in V$. Then $\sum_{u \in V} \deg_T(u) = 2n - 1$.

(c) Let $M = (S, \mathcal{I})$ be a matroid and $k$ a natural number. Define $\mathcal{I}' = \{I \in \mathcal{I} \mid |I| \leq k\}$. Then $M' = (S, \mathcal{I}')$ is a matroid.

(d) Given a directed graph $G = (V,E)$ with edge costs $c : E \to \mathbb{Q}$, one can determine in polynomial time whether $G$ contains a cycle of negative cost.

(e) Let $G = (V,E)$ be a directed graph with capacities $c : E \to \mathbb{Q}^+$ and source and destination nodes $s,t \in V$. If $f$ is a preflow and $\sum_{u \in V \setminus \{s,t\}} e_f(u) = 0$ then $f$ is a flow.

(f) Let $M^*$ be a maximum cardinality matching of an undirected graph $G = (V,E)$. If $M$ is a matching of $G$ and $k = |M^*| - |M| \geq 1$ then the shortest $M$-augmenting path contains at most $|M|/k$ edges from $M$.

(g) If $\Pi_1 \in P$ and $\Pi_2 \preceq \Pi_1$, then $\Pi_2 \in P$.

(h) If $\Pi_1$ is $NP$-complete and $\Pi_1 \preceq \Pi_2$, then $\Pi_2$ is $NP$-complete.

(i) The *metric TSP problem* is strongly $NP$-complete.

(j) The *minimum spanning tree problem* is a special case of the *Steiner tree problem*.

**Problem 2** (10 points). Consider the following problem:

***Allocation Problem***:

*Given*: A bipartite graph $G = (X \cup Y, E)$ with edge weights $w : E \to \mathbb{Q}^+$.

*Goal*: Find a subset $M \subseteq E$ such that each node in $X$ is incident to at most one edge in $M$ and the total weight $\sum_{e \in M} w(e)$ is maximized.

Show that the *allocation problem* can be solved by the greedy algorithm for matroids.

**Problem 3** (15 points). Consider the following problem:

***Hit-All-Shortest-Paths Problem***:

*Given*: A directed graph $G = (V,E)$ with edge costs $c : E \to \mathbb{Q}^+$, a source node $s \in V$ and a target node $t \in V$.

*Goal*: Find a subset $C \subseteq E$ of minimum cardinality such that every shortest $s,t$-path $P$ (with respect to $c$) is *hit* by $C$, i.e., $P \cap C \neq \emptyset$.

Derive an algorithm that solves this problem, prove its correctness and analyze its running time.

**Problem 4** (10 points). Consider the following problem:

***Set Cover Problem***:

*Given*: A set $U = \{1,\ldots,n\}$ of $n$ elements, a collection of $m$ subsets $S_1,\ldots,S_m \subseteq U$ and a natural number $K$.

*Goal*: Determine whether there is a selection of at most $K$ subsets such that their union is $U$, i.e., whether there exists some $I \subseteq \{1,\ldots,m\}$ with $|I| \leq K$ such that $\bigcup_{i\in I} S_i = U$.

Prove that the *set cover problem* is $NP$-complete.

**Problem 5** (10 points). Consider the decision variant of the *knapsack problem*:

***Knapsack Problem***:

*Given*: A set $N = \{1,\ldots,n\}$ of $n$ items with each item $i \in N$ having a profit $p_i \in \mathbb{Z}^+$ and a weight $w_i \in \mathbb{Z}^+$, a knapsack capacity $B \in \mathbb{Z}^+$ and a natural number $K$.

*Goal*: Determine whether there exists a subset $X \subseteq N$ such that $w(X) = \sum_{i\in X} w_i \leq B$ and $p(X) = \sum_{i\in X} p_i \geq K$.

Prove that the *knapsack problem* is $NP$-complete. (Hint: Use that the following *subset sum problem* is $NP$-complete: Given $n$ non-negative integers $s_1,\ldots,s_n$ and a parameter $L$, determine whether there is a subset of these numbers whose total sum is $L$.)

**Problem 6** ($15+10$ points). Consider the *multiway cut problem*:

***Multiway Cut Problem***:

*Given*: A connected, undirected graph $G = (V,E)$ with edge weights $w : E \to \mathbb{Q}^+$ and a set $T = \{t_1,\ldots,t_k\} \subseteq V$ of *terminals*.

*Goal*: Find a subset $E' \subseteq E$ of minimum total weight $w(E') = \sum_{e\in E'} w(e)$ whose removal disconnects all terminals in $T$ from each other.

(a) Derive a 2-approximation algorithm for the *multiway cut problem*. (Hint: Combine $k$ minimum weight cuts $C_1,\ldots,C_k$, where $C_i$ separates $t_i$ from the rest of the terminals.)

(b) Refine your algorithm to obtain a $(2-2/k)$-approximation algorithm and provide an example that shows that this approximation ratio is tight.