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Test: Linear Structures 1
201300056: Structures en Models
Friday, October 23, 2015; 13:45 - 15:15

This exam consists of 6 problems. All answers must be justified. A (graphical) calculator may be used only for checking your answers.

1. A linear transformation $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ is given by $T(f(x)) = (x-1)f''(x)$. Let β be the standard basis for $P_3(\mathbb{R})$ and γ the standard basis for $P_2(\mathbb{R})$.

(a) [6pt] Determine $[T]_\beta^\gamma$.

(b) [4pt] Verify that $[T(x^3+3x^2+3x+1)]_\gamma = [T]_\beta^\gamma[x^3+3x^2+3x+1]_\beta$.

2. A linear transformation $T : V \to W$ is not one-to-one but onto. Furthermore, $\dim(W) > 0$.

(a) [6pt] Prove that: (i) there exists a vector $u \neq \mathbf{0}$ such that $u \in N(T)$; (ii) there exists a vector $v \in V$, such that $v \notin N(T)$; (iii) if $S = \{u_1, u_2, \ldots, u_k\} \subseteq N(T)$ is a linearly independent set and $v \notin N(T)$, then the set $S \cup v = \{u_1, u_2, \ldots, u_k, v\}$ is linearly independent.

(b) [4pt] Prove that $\dim(W) < \dim(V)$.

(c) [4pt] $U : W \to Z$ is a linear transformation. Prove that the transformation $UT : V \to Z$ is not one-to-one.

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 4 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

(a) [6pt] Find the inverse of A.

(b) [4pt] Find the coordinates of the vector $(1,2,3)$ in the basis $\beta' = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, where $\mathbf{a}_i$, $i = 1,2,3$, is a column i of matrix A.

4. In (a),(b), the reduced row echelon form of the augmented matrix of a linear system $Ax = b$ is as follows:

$$\begin{pmatrix} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) [4pt] $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are columns of A. Compute A if

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

(b) [6pt] Find the solution set of this linear system.

5. [6pt] Let A be an $m \times n$ matrix and C be an $m \times m$ invertible matrix. Prove that the system $Ax = b$ has the same solution set as $(CA)x = Cb$.

6. [10pt] Determine the eigenvalues and one of the eigenvectors of the following matrix:

$$\begin{pmatrix} -1 & 2 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Tota: 60 points

NB: grade=([score Chapter 1] + [score at this test]+10)/10.