Linear Structures 1. 2015-201300056-1A: Structures and Models
Thursday 5 November 2015; 8:45 - 11:45 a.m.

This exam consists of 8 questions. All answers must be justified.
A (graphical) calculator may be used only for checking your answers.

1. [2pt] A set V consists of vectors (a_1, a_2) where $a_1, a_2 \in \mathbb{R}$. Addition and scalar multiplication are defined as follows:

$$(a_1, a_2) + (b_1, b_2) = (0, -a_1 - b_1), \quad c(a_1, a_2) = (ca_1, ca_2), \quad c \in \mathbb{R}.$$

Is V a vector space over $\mathbb{R}$? Justify your answer.

2. [3pt] Given is $S \subset M_{2\times 2}(\mathbb{R})$:

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \right\}.$$

Determine whether S is a basis for $M_{2\times 2}(\mathbb{R})$.

3. [4pt] Prove that a set of two or more vectors is linearly dependent if and only if one of these vectors can be written as a linear combination of the other vectors.

4. A transformation $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ is given by

$$T(f(x)) = xf(x) + f'(x), \quad f(x) \in P_2(\mathbb{R}).$$

(a) [2pt] Take $\beta = \{1, x, x^2\}$ as ordered basis for $P_2(\mathbb{R})$ and $\gamma = \{1, x, x^2, x^3\}$ as ordered basis for $P_3(\mathbb{R})$. Determine $[T]_\beta^\gamma$.

(b) [2pt] Verify that $[T(f(x))]_\gamma = [T]_\beta^\gamma [f(x)]_\beta$ for $f(x) = x^2 + 2x + 1$.

5. V and W are vector spaces, $T : V \to W$ is a linear transformation.

(a) [3pt] Prove that $N(T)$ is a subspace of V.

(b) [3pt] Prove that if T is one-to-one, then $\dim(V) \leq \dim(W)$.

see the other side

6. Consider a linear system $Ax = b$:

$$\begin{array}{rrrrrl} x_1 & +2x_2 & -x_3 & +3x_4 & & = 1 \\ -x_1 & +4x_2 & -5x_3 & +9x_4 & -6x_5 & = -6 \\ 2x_1 & +2x_2 & & +3x_4 & +2x_5 & = 4 \end{array}$$

(a) [4pt] Solve the system. Find the general solution of this system.

(b) [2pt] Determine the rank of A. Determine r linearly independent columns of A, where $r =$ rank (A).

(c) [2pt] Give an example of a 5×3 matrix B such that $B \neq O$ and $AB = O$, where O is a zero matrix.

7. [2pt] A, B are $n \times n$ matrices. Prove that if AB is invertible, then both A and B are invertible.

8. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & \alpha \\ 1 & 0 & 2.5 & 7 \\ -2 & 2 & 2 & 0 \end{pmatrix}.$$

(a) [3pt] Determine $\det(A)$.

(b) [2pt] For which α is it true that $\lambda = 0$ is an eigenvalue of A?

(c) [2pt] Is the vector $(1, -2, 2, 0)$ an eigenvector of A?

Total: 36 points

NB: mark=[[number of points]+4)]/4, rounded to a whole number.