Resit.
Linear Structures 1. 2015-201300056-1A: Structures and Models
Monday 4 January 2016

This exam consists of 10 questions. All answers must be justified.
A (graphical) calculator may be used only for checking your answers.

1. [5pt] Given is a set $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. The addition and scalar multiplication are defined as follows:

$$(a_1, a_2) + (b_1, b_2) = (2a_1 + 2b_1, a_2 + b_2), \quad c(a_1, a_2) = (ca_1, ca_2),\ c \in \mathbb{R}.$$

Prove that V is not a vector space by proving that one of the following properties is not satisfied:

(VS 7) For all $u, v \in V$, $c \in F$, $c(u + v) = cu + cv$.

(VS 8) For all $v \in V$, $c, d \in F$, $(c + d)v = cv + dv$.

2. [5pt] Give the definition of linear independence.

3. [10pt] Determine whether the set $\{(1, 2, 3, 1), (1, 3, 4, 2), (1, -1, 1, 2), (3, 2, 1, 3)\}$ is a basis for $\mathbb{R}^4$.

4. [5pt] $T : V \to W$ is a linear transformation. Prove that the null space $N(T)$ is a subspace of V.

5. [5pt] A linear transformation $T : V \to W$ is an isomorphism. Prove that $\dim(V) = \dim(W)$.

6. $P_2(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ is given by

$$T(ax^2 + bx + c) = \begin{pmatrix} a & b + c \\ b + c & -a \end{pmatrix}.$$

(a) [5pt] Which statement out of i)-iv) is true: i) T is one-to-one, ii) T is onto, iii) T is both one-to-one and onto, iv) T is neither one-to-one nor onto?

(b) [5pt] Determine $[T]_\beta^\gamma$ where β is the standard basis for $P_2(\mathbb{R})$ and γ is the standard basis for $M_{2\times2}(\mathbb{R})$.

(c) [5pt] Show that $[T(x^2 + 2x + 1)]_\gamma = [T]_\beta^\gamma [x^2 + 2x + 1]_\beta$.

7. [10pt] $Ax = b$ is a linear system, K is the set of solutions and K_H is a set of solutions to the homogeneous system $Ax = \mathbf{0}$. Prove that

$$K = \{s\} + K_H,$$

where s is a particular solution to $Ax = b$. (Here $S + U = \{s + u : s \in S, u \in U\}$).

8. Denote by $\underline{a}_j$, $j = 1, 2, 3, 4, 5$, column j of a 4×5 matrix A. Given is:

$$\underline{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -3 \end{pmatrix}, \ \underline{a}_2 = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 2 \end{pmatrix}, \underline{a}_4 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The reduced row echelon form of the augmented matrix $(A\ b)$ of the system $Ax = b$ is as follows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) [5pt] Write the solution to this system in a parametric vector form.

(b) [10pt] Compute the other two columns of matrix A and the vector b.

9. A is a 7×3 and B is a 3×7 matrix.

(a) [5pt] Determine the dimensions of AB. Show that each row of AB is a linear combination of the rows of B.

(b) [5pt] What can you tell about rank (AB)? Justify your answer.

10. [10pt] Consider the matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Determine the eigenvalues and the associated eigenvectors of matrix A.

Total: 90 points

NB: mark=([number of points]+10)/10.