Test Linear Structures 1
201300056: Structures en Models
Thursday November 10 2016; 8:45 - 11:45

This test contains 6 problems. A (graphical) calculator can be used only to check your answers. **IMPORTANT: Explain well how you obtained your answers. Motivate EACH of your answers.** A correct answer without clearly explained solution will give **at most 1pt** in each of the questions.

1. [6pt] $P_n(\mathbb{R})$ is the vector space of polynomials of degree at most n. Fix $a \in \mathbb{R}$ and consider a subset $W \subset P_n(\mathbb{R})$ of polynomials $f(t)$ such that $f(a) = 0$:

$$W = \{f(t) : f(a) = 0\}.$$

Prove that W is a subspace of $P_n(\mathbb{R})$.

2. [6pt] Prove that a set of vectors $\{v_1, v_2, \ldots, v_p\} \subset V$ is linearly dependent if and only if at least one of the vectors is a linear combination of the other vectors.

3. [6pt] Determine whether set S below is a basis for $M_2(\mathbb{R})$:

$$S = \left\{ \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \right\}.$$

4. [6pt] $T : V \to W$ is a linear transformation, and $\dim(V) > \dim(W)$. Prove that T cannot be one-to-one.

5. [6pt] $T : V \to W$ is a linear transformation, and $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for V. Prove that $\mathrm{span}(T(\beta)) = R(T)$.

6. A linear transformation $T : P_3(\mathbb{R}) \to \mathbb{R}^4$ is given by

$$T(ax^3 + bx^2 + cx + d) = \begin{pmatrix} a+b \\ b+c \\ c+d \\ d+a \end{pmatrix}.$$

(a) [5pt] Determine $[T]_\beta^\gamma$, where β and γ are the standard bases for $P_{\cancel{2}\,3}(\mathbb{R})$ and $\mathbb{R}^4$, repectively.

(b) [5pt] Show that $[T(x^3+2x^2+3x+1)]_\gamma = [T]_\beta^\gamma [x^3+2x^2+3x+1]_\beta$.

7. [6pt] Let A be an $n \times n$ matrix. Prove that a linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if the matrix A is invertible.

8. Let $\underline{a}_j$, where $j = 1,2,3,4,5$, denote the j-th column of a 3×5 matrix A. The reduced echelon form of the augmented matrix $(A|\mathbf{b})$ of the system $A\mathbf{x} = \mathbf{b}$ is given as follows:

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Further, the 1-st and the 4-th columns of A are given as follows:

$$\underline{a}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad \underline{a}_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

(a) [5pt] Determine $\mathrm{rank}(A)$ and $\dim(K_H)$, where K_H is a solution set of $A\mathbf{x} = \mathbf{0}$.

(b) [5pt] Give an example of a 5×2 matrix B such that $B \neq O$, but $AB = O$, where O is a zero matrix of a corresponding dimension.

(c) [5pt] Find the solution set of $A\mathbf{x} = \mathbf{b}$.

(d) [8pt] Compute the other three columns of A and the vector $\mathbf{b}$.

9. Matrix A and vector $\mathbf{b}$ are given as follows:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

(a) [5pt] Find the inverse of matrix A.

(b) [6pt] Find the coordinates of $\mathbf{b}$ in the basis, consisting of the columns of A.

10. [10pt] Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}.$$

Total: 90pt

grade=([number of points]+10)/10.