Test Chapter 1. LS 1.

30-09-2016

Solutions

1. $S = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$c(a_1, a_2) = (c|a_1|, ca_2)$$

S is not a vector space because e.g. (VS 5) does not hold.

(VS 5) $1 \cdot x = x \quad \forall x \in S$

$\forall a_1 < 0, \; 1 \cdot (a_1, a_2) = (|a_1|, a_2) \neq (a_1, a_2)$

2. $S \subseteq V$. 1. $S = \emptyset \Rightarrow \text{span}(S) = \{\underline{0}\}$ - subspace of V.

2. $\exists v \in S$, $v \in V$

a) $0 \cdot v = \underline{0} \in \text{span}(S)$

b) $x, y \in \text{span}(S) \Rightarrow$

$x = a_1 v_1 + \ldots + a_n v_n$ for some $v_1, \ldots, v_n \in S$, $a_1, \ldots, a_n \in F$

$y = b_1 w_1 + \ldots + b_m w_m$ for some $w_1, \ldots, w_n \in S$, $b_1, \ldots, b_m \in F$

$x + y = a_1 v_1 + \ldots + a_n v_n + b_1 w_1 + \ldots + b_m w_m \in \text{span}(S)$

c) $x \in \text{span}(S)$, as in b). Then $\forall c \in F$

$cx = ca_1 v_1 + \ldots + ca_n v_n \in \text{span}(S)$ ↗

a), b), c) hold $\Rightarrow$ span(S) is a subspace of V.

3. Take $v_1, \dots, v_n \in S_1 \cap S_2$ $(S_1 \cap S_2 = \emptyset \Rightarrow$ lin. indep.$)$

Consider the equation

$$a_1 v_1 + \dots + a_n v_n = \underline{0}$$

$v_1, \dots, v_n \in S_1$ - linear independent set

$\Rightarrow a_1 = \dots = a_n = 0 \quad \forall v_1, \dots, v_n \in S_1 \cap S_2$

$\Rightarrow S_1 \cap S_2$ is a linear independent set.

4. $\dim(P_3(\mathbb{R})) = 4, \quad |S| = 4.$

Hence, we need to check only one of the two properties: 1) S linearly independent

2) S generates $P_3(\mathbb{R})$

We will check whether S is linearly independent.

$$a_1(x^3+2x) + a_2(x^3+x^2+x) + a_3(x^2-x+1) + a_4(x^3+2x+2) = 0.$$

$$\begin{cases} a_1 + a_2 + a_4 = 0 \\ a_2 + a_3 = 0 \\ 2a_1 + a_2 - a_3 + 2a_4 = 0 \\ a_3 + 2a_4 = 0 \end{cases}$$

Augmented matrix: $\nearrow$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a_4 is a free variable, so we can choose $a_4 \neq 0$.

$\Rightarrow$ S is <u>not</u> linearly independent
$\Rightarrow$ S is <u>not</u> a basis for $P_3(\mathbb{R})$.

5. Linear system:

(1,1): $x_1 - x_2 + x_4 = 0$
(1,2): $x_2 + x_3 + x_4 = 4$
(2,1): $0 = 0$
(2,2): $-x_1 + 2x_2 + 2x_3 + x_4 = 2$

Augmented matrix:

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 4 \\ -1 & 2 & 2 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 & 2 \end{pmatrix} \to$$

$\nearrow$

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix}$$

x_4 - free variable

$x_3 = -2 - x_4$

$x_2 = 6$

$x_1 = 6 - x_4$. We found the solution

$\Rightarrow$

$\begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}$ is a linear combination of the matrices is S

Example of the coefficients: $6, 6, -2, 0$.

6. $\forall\ v \in V\ \exists$ unique $a_1 \dots a_n$ s.t. $v = a_1 u_1 + \dots + a_n u_n$.

Prove that $\beta = \{u_1, u_2, \dots, u_n\}$ is a basis for V.

Proof:

1) $\forall\ v \in V$, we have $v \in \operatorname{span}(\beta)$
$\Rightarrow V \subseteq \operatorname{span}(\beta) \Rightarrow V = \operatorname{span}(\beta)$

2) We need to prove that β is linearly indep.
$a_1 u_1 + a_2 u_2 + \dots + a_n u_n = \underline{0}$ $(*)$

$a_1 = \dots = a_n = 0$ satisfy $(*)$ and by the conditions $a_1, \dots, a_n$ are unique $\Rightarrow$ β is linearly independent.