Linear Structures 1

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Solutions

1. $T: V \to W$ lin. transformation.

a) $T(\underline{0}) = \underline{0} \Rightarrow \underline{0} \in R(T)$

b) $x, y \in R(T) \Rightarrow \exists\ u, v \in V$ such that

$T(u) = x, \quad T(v) = y$

$x + y = T(u) + T(v) = T(u+v)$

$u + v \in V \Rightarrow T(u+v) \in R(T)$

$\Rightarrow x + y \in R(T)$

c) $x \in R(T) \Rightarrow \exists\ u \in V$ such that $T(u) = x$

$c \in F$

$cx = c\,T(u) = T(cu)$

$cu \in V \Rightarrow T(cu) \in R(T)$

$\Rightarrow cx \in R(T)$

a), b), c) hold $\Rightarrow R(T)$ is a subspace of W.

2. a) Example:

$T(ax^2 + bx + c) = ax^2 + ax + a$

Using a)

b) $[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$T(1) = 0$

$T(x) = 0$

$T(x^2) = \cancel{a}\,x^2 + x + 1$

c) $\left([T]_\beta^\beta\right)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = [T]_\beta^\beta$

Without using a)

b) $T(f(x)) = x f'(x)$

$T(1) = 0$

$T(x) = x$

$T(x^2) = 2x^2$

$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

c) $T^2(f(x)) = x\left(x f'(x)\right)'$

$= x\left(f'(x) + x f''(x)\right)$

$= x f'(x) + x^2 f''(x)$

$T^2(1) = 0$

$T^2(x) = x$

$T^2(x^2) = 4x^2$

$[T^2]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

3. $T: V \to W$ lin. transf. one-to-one, not onto

$\beta = \{u_1, u_2, \dots, u_n\}$ - basis for V.

a) $T(\beta) = \{T(u_1), T(u_2), \dots, T(u_n)\}$

~~Suppose~~ Consider the equation

$$a_1 T(u_1) + a_2 T(u_2) + \dots + a_n T(u_n) = \underline{0}. \quad (*)$$

Since T is a lin. transformation it follows from $(*)$ that

$$T(a_1 u_1 + a_2 u_2 + \dots + a_n u_n) = \underline{0}.$$

$\Rightarrow a_1 u_1 + a_2 u_2 + \dots + a_n u_n \in N(T)$

T is one-to-one $\Rightarrow N(T) = \{\underline{0}\}$.

$\Rightarrow a_1 u_1 + a_2 u_2 + \dots + a_n u_n = \underline{0}$

However, $\{u_1, u_2, \dots, u_n\} = \beta$ is linearly independent $\Rightarrow$

$$a_1 = a_2 = \dots = a_n = 0.$$

We conclude that $(*)$ has only a trivial solution $\Rightarrow T(\beta)$ is linearly independent.

b) Dimension theorem:

$$\dim(V) = \dim(N(T)) + \dim(R(T))$$

T one-to-one $\Rightarrow \dim(N(T)) = 0$

T not onto $\Rightarrow R(T) \neq W$, $\dim(R(T)) < \dim(W)$.

Hence,

$$\dim(V) = 0 + \dim(R(T)) < \dim(W).$$

4.

a)
$$\begin{pmatrix} 1 & -1 & 0 & 3 & 2 & 0 \\ 2 & -2 & 1 & 8 & 0 & 4 \\ -1 & 1 & -3 & -9 & 1 & -3 \end{pmatrix} \overset{\substack{r2 \leftarrow r2 - 2\times r1 \\ r3 \leftarrow r3 + r1}}{\sim} \begin{pmatrix} 1 & -1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 2 & -4 & 4 \\ 0 & 0 & -3 & -6 & 3 & -3 \end{pmatrix}$$

$$\overset{r3 \leftarrow r3 + 3\times r1}{\sim} \begin{pmatrix} 1 & -1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 2 & -4 & 4 \\ 0 & 0 & 0 & 0 & -9 & 9 \end{pmatrix} \overset{r3 \leftarrow r3/(-9)}{\sim} \begin{pmatrix} 1 & -1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\overset{\substack{r1 \leftarrow r1 - 2\times r3 \\ r2 \leftarrow r2 + 4\times r3}}{\sim} \begin{pmatrix} 1 & -1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$x_5 = -1$ $\quad x_3 = -2x_4$ $\quad x_1 = 2 - 3x_4 + x_2$

x_4 - free $\quad x_2$ - free $\quad x_2 = t \in \mathbb{R}$, $x_4 = s \in \mathbb{R}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2+t-3s \\ t \\ -2s \\ s \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

$s, t \in \mathbb{R}$

$$K = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \; s, t \in \mathbb{R} \right\}$$

b) The reduced echolon form of A has three lin. independent columns => same is true for the matrix A => rank(A) = 3 (because row operations are rank preserving)

c) $K_H = \left\{ t\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \; s, t \in \mathbb{R} \right\}$ because

$K = \{s\} + K_H$, where s is a particular solution of $Ax = b$.
In this case $s = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$.

$\dim(K_H) = 2$ because $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ are

lin. indep. $\Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis of K_H.

Alternatively: $\dim(K_H) = 5 - \operatorname{rank}(A)$
$= 5 - 3 = 2$.

5. AB invertible. Let $C = (AB)^{-1}$

We have: $C(AB) = I$
$(CA)B = I$
$\Rightarrow (CA)$ is an inverse of $B \Rightarrow$
$\Rightarrow B$ is invertible.

Also: $(AB)C = I$
$A(BC) = I$

$\Rightarrow BC$ is an inverse of $A \Rightarrow$
$\Rightarrow A$ is invertible.

6. a) λ - eigenvalue of $A \Rightarrow \exists$ a non-zero vector x s.t.
$Ax = \lambda x$.
Then $Ax - \lambda x = \underline{0}$
$Ax - \lambda I x = \underline{0}$
$(A - \lambda I)x = \underline{0}$ holds for some $x \neq 0$

$\Rightarrow (A - \lambda I)x = \underline{0}$ has a non-trivial solution

$\Rightarrow$ rank $(A - \lambda I) < n$, $(A - \lambda I)$ is not invertible

$\Rightarrow \det(A - \lambda I) = 0.$

6) $\begin{vmatrix} 1-\lambda & 2 & 0 \\ -2 & 6-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\Big((1-\lambda)(6-\lambda)+4\Big)$

$= (3-\lambda)(\lambda^2 - 7\lambda + 10)$

$\lambda_{1,2} = \dfrac{7 \pm \sqrt{49-40}}{2} = \dfrac{7 \pm 3}{2}$

$\lambda_1 = 5, \quad \lambda_2 = 2$

Eigenvalues of the given matrix are:
$\lambda = 3, 5, 2.$