Exam: Linear Structures 1.
Applied Mathematics, 2018-1A: Structures and Models
October 30 2018; 8:45 - 11:45

This exam consists of 9 problems.
A (graphical) calculator is not needed and is **not allowed** at the exam.

The following is IMPORTANT and will be taken into account for grading:
i) Define all variables and explain notations that you introduce in your solution.
ii) Clearly explain each step of your solution, in words or by a clear derivation.

1. [10pt] Let $T : V \to W$ be a linear transformation. Prove that $N(T)$ is a subspace of V.

2. Consider a set

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

(a) [5pt] Prove that β' is a basis of $\mathbb{R}^3$.

(b) [10pt] Find the coordinates in the basis β' for the vectors from the standard basis:

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

3. [5pt] Let V, W be linear spaces with finite dimensions. Let $T : V \to W$ be a linear transformation. Assume that T is onto. Prove that $\dim(W) \leq \dim(V)$.

4. $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ is given by

$$T((f(t))) = (t+1)f'(t), \quad t \in \mathbb{R}.$$

(a) [5pt] Determine $[T]_\beta^\beta$ where β is the standard basis for $P_3(\mathbb{R})$. Check your answer by verifying that $[T(f(t))]_\beta = [T]_\beta^\beta [f(t)]_\beta$ for $f(t) = t^3 + 3t^2 + 3t + 1$.

(b) [5pt] Is T one-to-one? Is T onto?

SEE OTHER SIDE

5. [10pt] $T : V \to W$ is a linear transformation. Assume that for some set

$$S = \{u_1, u_2, \dots, u_m\} \subset V$$

it holds that the set $\{T(u_1), T(u_2), \dots, T(u_m)\} \subset W$ is linearly independent. Prove that S is linearly independent.

6. The reduced row echelon form of the augmented matrix $(A \,|\, \mathbf{b})$ of the system $A\mathbf{x} = \mathbf{b}$ is the matrix $(B \,|\, \mathbf{c})$ given below:

$$(B \,|\, \mathbf{c}) = \left(\begin{array}{ccccc|c} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right).$$

(a) [5pt] Determine the solution set of the linear system $A\mathbf{x} = \mathbf{b}$.

(b) [5pt] Give an example of a 5×2 matrix C such that $AC = O$, where O is the matrix consisting of zeros, and $\mathrm{rank}(C) = 2$.

(c) [5pt] Denote by $\mathbf{a}_j$ column j of matrix A, and denote by $\mathbf{b}_j$ column j of matrix B, where $j \in \{1, 2, 3, 4, 5\}$. Prove that $\mathbf{b}_4 = -\mathbf{b}_1 + 2\mathbf{b}_3$ implies that $\mathbf{a}_4 = -\mathbf{a}_1 + 2\mathbf{a}_3$.

7. [5pt] Let A and B be $n \times n$ matrices. Prove that if AB is invertible then both A and B are invertible.

8. [10pt] Compute the determinant of matrix A below using elementary row and/or column operations:

$$A = \begin{pmatrix} -1 & -2 & 1 & 2 \\ -1 & 5 & 3 & 1 \\ 1 & 3 & 0 & -1 \\ -2 & 0 & 2 & 2 \end{pmatrix}.$$

9. (a) [5pt] Recall that $\lambda \in \mathbb{R}$ is an eigenvalue of an $n \times n$ matrix A if $A\mathbf{x} = \lambda\mathbf{x}$ for some non-zero vector $\mathbf{x} \in \mathbb{R}^n$. Prove that each eigenvalue λ of an $n \times n$ matrix A must satisfy the characteristic equation $\det(A - \lambda I) = 0$.

(b) [5pt] Prove that a square matrix is *not* invertible if one of its eigenvalues equals zero.

Total: 90 points

NB: grade=([number of points]+10)/10.