Exam: Linear Structures 1. PART 1
Applied Mathematics, 2021-1A: Structures and Models
November 7 2022; 8:45 - 10:15

This exam consists of 10 problems which are divided into two parts:
Grasple (digital): 8 problems
Open Questions (written): 2 problems.

Grasple
Enter your answers in Grasple in the required form. Follow the instructions precisely.
For the statements, you choose one of three options: true (T), false (F), or no answer (N).
For each correct T or F you will receive (partial) points.
One incorrect T or F results in zero points for that entire question.
Total score for Grasple: 50 points.
Required score: 25 points.

Open Questions
Write the solutions following the four steps.
Step 1. State the important information and summarize the problem.
Step 2. Devise a plan.
Step 3. Execute the plan.
Step 4. Analyze your solution and the answer.
Think, for example, of the following questions:
– Can you interpret the solution and/or the result intuitively?
– Does the solution/result make sense in special cases?
– Which role each condition played in the solution?
– Could you relax some assumptions of the problem?
– Is the problem related to other problems or results?
Total score for Open Questions: 40 points.
Steps 1+2: 40%, Step 3: 40%, Step 4: 20%.
Required score: 20 points.

A (graphical) calculator is not needed and is **not allowed** at the exam.

PART 1: Grasple questions

1. [8pt] Which of the following sets are subspaces of the corresponding vector spaces? Indicate 'true' (T) if the set is a subspace. Otherwise, indicate 'false' (F), or give no answer (N).

Instructions: *Write your answers as a string: e.g.* ***TFFN*** *means the first subset is a subspace, sets 2 and 3 are not a subspace and you give no answer to statement 4.*

F (1) The set of all polynomials in $P_3(F)$ with odd coefficients

T (2) The set of all vectors in $\mathbb{R}^3$ of the form $\begin{pmatrix} a+1 \\ b \\ c \end{pmatrix}$ for $a, b, c \in \mathbb{R}$.

F (3) The set of all vectors in $\mathbb{R}^2$ of the form $\begin{pmatrix} x \\ y \end{pmatrix}$ where $y = x + 1$.

F (4) The set of all real-valued functions $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $f(1) = 1$.

2. [5pt] For each of the following statements, indicate whether the statement is true (T) or false (F), or give no answer (N).

Instructions: *Write your answers as a string: e.g.* ***TFFN*** *means the first statement is true, statements 2 and 3 are false and you give no answer to statement 4.*

T (1) The set $\{1, x+3, x^2-2, x^2+x\}$ is a linearly dependent subset of the space of polynomials.

F (2) If S is a linearly dependent subset of V, then $|S| > \dim(V)$.

T (3) If S is a linearly dependent subset of V then each set containing S is linearly dependent.

T (4) If S is a linearly independent subset of V, then $|S| < \dim(V)$.

3. [5pt] For each of the following statements, indicate 'true' (T) if the statement holds for **any** matrix $A \in M_{5\times 3}(F)$ and **any** matrix $B \in M_{3\times 5}(F)$. Otherwise, indicate 'false' (F) or give no answer (N).

Instructions: *Write your answers as a string: e.g.* ***TFFN*** *means the first statement is true, statements 2 and 3 are false and you give no answer to statement 4.*

F (1) rank$(A) = 3$.

T (2) rank$(AB) \leq$ rank(B).

F (3) The system $B\mathbf{x} = \mathbf{b}$ has infintely many solutions for any given $\mathbf{b} \in \mathbb{R}^3$. no solution

T (4) $R(L_A) \subseteq \mathbb{R}^5$.

4. [5pt] The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $T(a,b,c) = (a+b, 3c-a)$. Consider two ordered basis α and β for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively:

$$\alpha = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}, \quad \beta = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}.$$

Write down $[T]_\alpha^\beta$.

Instructions: *Use the tool in Grasple to create a matrix of the correct size and fill out its elements.*

5. [5pt] Given is the matrix $A \in M_{3\times 3}$:

$$A = \begin{pmatrix}1 & 2 & 3\\0 & 1 & 4\\5 & 6 & 0\end{pmatrix}$$

Find matrix B such that $BA = I_3$.

Instructions: *Use the tool in Grasple to create a matrix of the correct size and fill out its elements.*

6. [7pt] Given is the following set W of vectors in $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix}2\\-3\\4\end{pmatrix}, \begin{pmatrix}-1\\1\\2\end{pmatrix}, \begin{pmatrix}-4\\6\\-8\end{pmatrix}, \begin{pmatrix}0\\-3\\-18\end{pmatrix}, \begin{pmatrix}1\\0\\-2\end{pmatrix} \right\}$$

Find the basis of $\mathbb{R}^3$ consisting of vectors in W.
Instructions: *Give your answer as a set of vectors.*

7. [5pt] The determinant of $A = \begin{pmatrix}a & 1 & 2 & 3\\b & 4 & 5 & 6\\c & 7 & 8 & 9\\d & 10 & 11 & 12\end{pmatrix}$ is equal to 3. Compute the determinant of matrix B:

$$B = \begin{pmatrix}2a & 2b & 2c & 2d\\2 & 5 & 8 & 11\\1 & 4 & 7 & 10\\3-2a & 6-2b & 9-2c & 12-2d\end{pmatrix}.$$

8. [10pt] The reduced echelon form of the augmented matrix $A\mathbf{x} = \mathbf{b}$ is given as follows:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 & -3 \end{pmatrix}$$

a) Find the solution set for the system $A\mathbf{x} = \mathbf{b}$.

Instructions: *Give your answer in vector form.*

b) What is the rank of A?

c) What is the dimension of the null space of A?

Exam: Linear Structures 1. PART 2
Applied Mathematics, 2021-1A: Structures and Models
November 7 2022; 13:45 - 15:15

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PART 2: Open questions.

1. [20pt] Let V be a vector space. Prove that $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for V if and only if for each $v \in V$ there exist unique $a_1, a_2, \ldots, a_n \in F$ such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

2. [20pt] $T : V \to W$ is a linear transformation. We know that T is onto and T is not one-to-one.

(a) Prove that $\dim(V) > \dim(W)$.

(b) Does $\dim(V) > \dim(W)$ imply that T is onto? Why or why not?

(c) Does $\dim(V) > \dim(W)$ imply that T is not one-to-one? Why or why not?