Exam Linear Structures II, 201700140.

Date	:	Februari 1, 2018
Place	:	Sportcentrum
Time	:	08.45 – 11.45

All answers must be motivated.
The use of (Scientific) calculator, formula sheet, or notes is not allowed.

1. Given the (complex) linear space spanned by $\{\sin(x), \cos(x), e^{2x}\}$. On this space we define the linear mapping T as the derivative, i.e.,

$$T(f) = \frac{df}{dx}.$$

   (a) Determine the eigenvalues and eigenvectors of T.

   (b) Prove that T is diagonisable.

   (c) Determine the eigenvalues of the inverse of T.

2. Let Z be the linear space of 2 by 2 complex matrices. On this space we define the following (candidate) inner product

$$\left\langle \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \right\rangle = (A_{11} + A_{12})\overline{(B_{11} + B_{12})} + \\ 2(A_{12} + A_{21})\overline{(B_{12} + B_{21})} + 3A_{21}\overline{B_{21}} + 4A_{22}\overline{B_{22}}. \qquad (1)$$

   (a) Show that (1) defines an inner product on Z.

   (b) Let W be the linear subspace spanned by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Determine an orthonormal basis of W.

   (c) Determine a non-zero element in $W^{\perp}$.

3. Let V be a linear space, and let W and Y be two linear subspaces of V. Furthermore, let T be an operator on V. We assume that both $W$and Y are T-invariant.

   (a) Prove that $W \cap Y$ is T-invariant.

   (b) Assume now that W is one-dimensional and is spanned by w. Show that w is an eigenvector of T.

4. Let A be an $n \times n$ matrix with characteristic polynomial $p(t) = (-1)^n(t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0)$.

   Prove that A is invertible if and only if $a_0 \neq 0$.

Z.O.Z.

5. Let V be a finite-dimensional complex inner product space, and let Q be a normal operator on V.

   (a) Prove that for every $\lambda \in \mathbb{C}$ the operator $Q + \lambda I$ is normal.

   (b) Prove that the null spaces of Q and Q^* are the same, i.e., prove that $N(Q) = N(Q^*)$.

   (c) Prove that if v is an eigenvector of Q associated to the eigenvalues λ, then v is an eigenvector of Q^*, but now with eigenvalue $\overline{\lambda}$.

6. Check whether the following statements are true or false. If true, provide a proof. If false, give a proof or counter example.

   (a) Let Q be a normal operator on the complex inner product space V which is invertible. Then Q is unitary.

   (b) Let S be a normal operator on the complex inner product space V with all eigenvalues real. Then S is self-adjoint.

   (c) If U and R are unitary operators on the complex inner product space V, then $U^{-1}R$ is unitary.

   (d) Let T be the operator from Exercise 1. There exists an inner product on the space spanned by $\{\sin(x), \cos(x), e^x\}$ such that T becomes self-adjoint.

Points[1]

Ex. 1		Ex. 2		Ex. 3		Ex. 4	Ex. 5		Ex. 6	
a	8	a	8	a	4	6	a	4	a	5
b	4	b	8	b	5		b	6	b	5
c	5	c	4				c	6	c	5
									d	5

[1] Total is 100. You get 10 points for free