Test Linear Structures II

Date : December 15, 2020
Time : 09.00 – 12.00

All answers must be motivated.
The use of (Scientific) calculator, formula sheet, or notes is not allowed.

1. For $a \in \mathbb{R}, b \in \mathbb{R}$, consider matrix $A = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$

   (a) Determine the eigenvalues of A.

   (b) For which values of a and b is A diagonisable?

2. Consider the following matrix

$$A = \begin{bmatrix} 1 & A_{12} & A_{13} & A_{14} \\ 0 & 1 & A_{23} & A_{24} \\ 0 & 0 & 1 & A_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   We consider the powers of A, and define $B = A^2$, $C = A^3$, and $D = A^4$. Furthermore, of a matrix Q, we denote the (i,j)-th element by Q_{ij}.

   Prove that

$$6B_{14} + D_{14} = 4A_{14} + 4C_{14}.$$

3. Let V be a complex inner product space and let T be a linear mapping from V to V. Prove that if the subspace W is T-invariant, then $W^{\perp}$ is T^*-invariant.

4. Consider the set of polynomials of degree 2 or lower with complex coefficients, $\mathbb{P}_2(\mathbb{C})$. On this space we define the (candidate) inner product

$$\langle p, q\rangle = p(0)\overline{q(0)} + \int_0^1 p'(x)\overline{q'(x)}dx. \qquad (1)$$

   where $q'(x)$ denotes the derivative of $q(x)$

   (a) Show that (1) defines an inner product on $\mathbb{P}_2(\mathbb{C})$.

   (b) Let $W \subset \mathbb{P}_2(\mathbb{C})$ be the subspace containing all constant polynomials. Determine (with respect to the inner product (1)) the orthogonal complement of W.

   (c) Does the following define an inner product on $\mathbb{P}_2(\mathbb{C})$

$$\langle p, q\rangle = \overline{p(0)}q(0) + \int_0^1 \overline{p'(x)}q'(x)dx?$$

Z.O.Z.

5. Consider as in the previous exercise the set of polynomials of degree 2 or lower with complex coefficients, $\mathbb{P}_2(\mathbb{C})$. On this space we define the inner product as given in (1). Let Z be the subspace consisting of polynomials of degree 1 or lower.

   (a) Construct an orthonormal basis of Z.

   (b) Let $q(x) = x^2$. Determine the orthogonal projection of q on Z.

6. On the inner product space of the previous two exercises, we define the following operator

$$(S(p))(x) = p(-x).$$

   Thus S mirrors the function in the vertical axis.

   Is S self-adjoint?

7. Check whether the following statements are true or false. If true, provide a proof. If false, give a proof or counter example.

   (a) If matrix A is unitarily equivalent to matrix B, then B is unitarily equivalent to A.

   (b) Any operator T for which $T = T^2$ (projection) has only the eigenvalues 0 and 1.

   (c) If the $n \times n$ matrix A has as only eigenvalue -1, then A is unitary.

   (d) Consider an operator T on an inner product space V. If $T(v) = -v$ for all $v \in V$, then T is normal for any inner product on V.

Points[1]

Ex. 1		Ex. 2	Ex. 3	Ex. 4		Ex. 5		Ex. 6	Ex. 7	
a	2	8	8	a	8	a	6	8	a	6
b	10			b	6	b	6		b	6
				c	4				c	6
									d	6

[1] Total is 100. You get 10 points for free