Resit Linear Structures II

Date : January 27, 2021
Time : 13.45 – 16.45

All answers must be motivated.
The use of (Scientific) calculator, formula sheet, or notes is not allowed.

1. Consider matrix $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & -2 & 1 \\ -1 & 1 & 1 & 1 \\ 2 & -2 & 0 & 0 \end{bmatrix}$
   Without computing the characteristic polynomial of A, find all $a \in \mathbb{R}$ for which $(1, 1, 1, a)$ is an eigenvector of A.

2. Consider the vector space of polynomials of degree 2 or lower with real coefficients, $\mathbb{P}_2(\mathbb{R})$, operator $T(f(x)) = f''(x) + f'(x) + f(0) \cdot x^2$, and vector $g(x) = x$.
   (a) Determine the T-cyclic subspace generated by $g(x)$.
   (b) Determine the characteristic polynomial of T without constructing a representation matrix of T.
   (c) Either prove, or find a counterexample to the following statement:
   For any vector $h(x)$ in the T-cyclic subspace generated by $g(x)$, the T-cyclic subspace generated by $h(x)$ equals the T-cyclic subspace generated by $g(x)$.

3. Consider complex vector space V with inner products $\langle .,. \rangle_1$ and $\langle .,. \rangle_2$.
   Also consider functions $\langle \mathbf{u}, \mathbf{v} \rangle_3 = i \cdot \langle \mathbf{u}, \mathbf{v} \rangle_1$ and $\langle \mathbf{u}, \mathbf{v} \rangle_4 = \langle \mathbf{u}, \mathbf{v} \rangle_1 - \langle \mathbf{u}, \mathbf{v} \rangle_2$ for all $\mathbf{u}, \mathbf{v} \in V$.
   For each of the following statements, either give a proof or a counterexample.
   (a) $\langle .,. \rangle_3$ defines an inner product on V.
   (b) $\langle .,. \rangle_4$ defines an inner product on V.

Z.O.Z.

4. Let Z be the linear space of 2 by 2 real matrices. On this space we define the following inner product:

$$\left\langle \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \right\rangle$$
$$= A_{11}B_{11} + (A_{11} + A_{12})(B_{11} + B_{12}) + (A_{11} + A_{21})(B_{11} + B_{21}) + A_{22}B_{22}.$$

   (a) Let W be the linear subspace spanned by $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Determine an orthonormal basis of W.

   (b) Determine $W^{\perp}$.

5. On the inner product space of the previous exercise, we define the following operator $T(A)$ that swaps the columns of A. That is: $T\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) = \begin{bmatrix} A_{12} & A_{11} \\ A_{22} & A_{21} \end{bmatrix}$.

   Is T normal?

6. Check whether the following statements are true or false. If true, provide a proof. If false, give a proof or counter example.

   (a) If an operator T on inner product space V is normal, then any basis for V of eigenvectors of T is orthogonal.

   (b) Let T be a normal operator on a complex inner product space. If $T^3 = I$, then $T = I$.

   (c) If U and R are unitary operators on the complex inner product space V, then $U^{-1}R$ is unitary.

   (d) Every linear operator on a one-dimensional inner product space is normal.

Points[1]

Ex. 1	Ex. 2		Ex. 3		Ex. 4		Ex. 5	Ex. 6	
10	a	6	a	8	a	7	8	a	6
	b	6	b	8	b	7		b	6
	c	6						c	6
								d	6