Test Linear Structures II

Date : December 21, 2020
Time : 08.45 – 11.45

All answers must be motivated.
The use of (Scientific) calculator, formula sheet, or notes is not allowed.

1. Consider an invertible linear operator T with eigenvalue $\lambda \neq 0$.

   (a) Prove that $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

   (b) Use mathematical induction on n to prove that λ^n is an eigenvalue of T^n for all $n \in \mathbb{N}$. (Recall that $0 \notin \mathbb{N}$).

2. Consider $C(\mathbb{R})$, the space of all continous real-valued functions defined on $\mathbb{R}$ and let $T(f) = f'$ denote the differentiation operator.

   (a) Determine a 1-dimensional T-invariant subspace that does **not** contain $g(x) = 1$.

   (b) Determine a 2-dimensional T-invariant subspace that contains $f(x) = x$.

   (c) Determine a 3-dimensional T-invariant subspace that contains $f(x) = x$.

   (d) Prove that the subspace found in b) is the **only** 2-dimensional T-invariant subspace that contains $f(x) = x$.

3. Consider $M_{2\times 2}(\mathbb{C})$, the space of 2×2 matrices with complex entries. On this space we define the following functions:

$$
\begin{aligned}
\langle A, B\rangle_1 &= \operatorname{tr}\left(B^t A\right) \\
\langle A, B\rangle_2 &= A_{11}\overline{B_{11}} + A_{22}\overline{B_{22}} \\
\langle A, B\rangle_3 &= |A_{11}B_{11}| + |A_{22}B_{22}| + |A_{12}B_{12}| + |A_{21}B_{21}|
\end{aligned}
$$

For each of these functions, show that it is **not** an inner product.

PTO

4. Consider $\mathbb{R}^3$ with inner product

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = (x_1 + x_2 + 2x_3)(y_1 + y_2 + 2y_3) + x_2 y_2 + x_3 y_3.$$

Let Z be the subspace consisting of all vectors $(a, 0, b), a, b \in \mathbb{R}$.

(a) Construct an orthonormal basis for Z.

(b) Let $\mathbf{v} = (0, 1, 0)$. Determine the orthogonal projection of $\mathbf{v}$ on Z.

5. On the inner product space of the previous exercise, we define the left-multiplication operator L_A, where

$$A = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Is L_A normal?

6. Check whether the following statements are true or false. If true, provide a proof. If false, give a proof or counterexample.

(a) Consider an operator T on a real vector space V for which $\langle T(\mathbf{v}), \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$. Then T is the zero-operator T_0.

(b) Consider a self-adjoint operator T for which $T^3 = T_0$. Then T is the zero-operator T_0 itself.

(c) Consider an operator T on an inner product space V. If $T(\mathbf{v}) = -\mathbf{v}$ for all $\mathbf{v} \in V$, then T is normal for any inner product on V.

(d) Consider an orthogonal operator T on finite-dimensional inner product space V. Then there exists a basis β for V for which $|\det([T]_\beta)| = 1$.

Points[1]

Ex. 1		Ex. 2		Ex. 3	Ex. 4		Ex. 5		Ex. 6	Ex. 7	
a	6	a	1	8	a	6	a	6	8	a	6
b	6	b	2		b	6	b	6		b	6
		c	2		c	6				c	6
		d	3							d	6

[1] Total is 100. You get 10 points for free