Resit Linear Structures II

Date : February 2, 2022
Time : 13.45 – 16.45

All answers must be motivated.
The use of (Scientific) calculator, formula sheet, or notes is not allowed.

1. Consider matrix $A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 2 & 1 & 3 & 4 \\ 1 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \end{bmatrix}$.

   Let $A = PDP^{-1}$,

   where D is a diagonal matrix, and the first column of P equals $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

   Determine the first column of D.
   Hint: It is not necessary to compute P or D.

2. Consider operator T on 2-dimensional vector space V. Prove the following statement: Either $T = cI$ for some scalar c, or there exists some vector $\mathbf{v} \in V$ for which V is the T-cyclic subspace generated by $\mathbf{v}$.

3. Consider the following candidate inner products on $\mathbb{C}^2$:

   $$\langle \mathbf{u}, \mathbf{v} \rangle_0 = 0 \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{C}^2$$
   $$\langle \mathbf{u}, \mathbf{v} \rangle_1 = 1 \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{C}^2$$

   (a) Prove that $\langle \mathbf{u}, \mathbf{v} \rangle_1 = 1$ is not an inner product on any subspace of $\mathbb{C}^2$.

   (b) Prove that $\langle \mathbf{u}, \mathbf{v} \rangle_0$ is an inner product on $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. (The subspace of $\mathbb{C}^2$ that contains only the zero-vector)

   (c) Prove that $\langle \mathbf{u}, \mathbf{v} \rangle_0$ is not an inner product on any subspace of $\mathbb{C}^2$ other than $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

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4. Check whether the following statements are true or false. If true, provide a proof. If false, give a proof or counterexample. Any proofs are not allowed to reference Theorem 6.4 (which states that Gram-Schmidt yields an orthonormal basis).

S1) For any inner product space V, for any vectors $\mathbf{u}, \mathbf{v} \in V$:
$\mathbf{v}$ and $\mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2} \mathbf{v}$ are orthogonal.

S2) For any inner product space V, for any vectors $\mathbf{u}, \mathbf{v} \in V$:
$\mathbf{v}$ and $\mathbf{u} - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{||\mathbf{v}||^2} \mathbf{v}$ are orthogonal.

5. Consider the vector space $V = \text{span}\{\sin(x), \cos(x)\}$ and differential operator $T(f) = f'$. Show that T is not self-adjoint under any inner product on V.

6. Check whether the following statements are true or false. If true, provide a proof. If false, give a proof or counterexample.

(a) The zero-operator T_0 is normal on any finite-dimensional inner product space.

(b) Let T be a self-adjoint operator on a finite-dimensional inner product space. If $T^3 = I$, then $T = I$.

(c) Let U and T be linear operators. If UT is unitary, then U and T are unitary.

(d) If matrix A is unitarily equivalent to matrix B, then B is unitarily equivalent to A.

Points[1]

Ex. 1	Ex. 2	Ex. 3		Ex. 4		Ex. 5	Ex. 6	
8	12	a	7	a	7	10	a	6
		b	8	b	7		b	6
✓		c	7				c	6
							d	6

[1]Total is 100. You get 10 points for free