Course : **Mathematics β1, Bernoulli**

Date : November 10, 2017
Time : 13.45 – 16.45 hrs

Motivate all your answers.
The use of electronic devices is not allowed.

1. [3pt] Solve the initial value problem

$$\begin{cases} y' + (3x^2 - 1)y = 0, \\ y(1) = 1. \end{cases}$$

2. Define $z = \dfrac{2i}{1 - i\sqrt{3}}$

   (a) [2 pt] Find the modulus (absolute value) and the argument of z.

   (b) [2 pt] Does there exist a $n \in \mathbb{N}$, such that $z^n = i$? Prove your claim. If such an n exists, determine the smallest possible value of n.

3. [5 pt] Solve the given initial-value problem

$$\begin{cases} y'' + y = x^2, \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

4. Consider the points $P(2, \sqrt{3}, 3)$ and $Q(2, -\sqrt{3}, -3)$. As usual $O(0, 0, 0)$.

   (a) [2 pt] Find the vector of length 1 in the direction of $\overrightarrow{QP}$.

   (b) [2 pt] Find the angle $\angle QOP$ (the angle between the line segments OP and OQ).

   (c) [2 pt] Compute the vector projection of $\overrightarrow{OQ}$ onto the straight line through points O and P.

5. Given are the three points $P(1,1,0)$, $Q(0,2,1)$ and $R(3,2,-1)$ in $\mathbb{R}^3$.

Let V be the plane that passes through P, Q and R and let ℓ_α be the line through the origin in the direction of vector $\mathbf{v} = (\alpha, 2, -6)$, for some $\alpha \in \mathbb{R}$

(a) [3 pt] Give the equation of V and determine the value of α such that $\ell_\alpha \perp V$.

(b) [2 pt] Prove or disprove the following statement:

$$\exists \alpha \in \mathbb{R}(\ell_\alpha \subset V).$$

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} \dfrac{\sin(\sin x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

(a) [3 pt] Show that f is continuous in 0.

(b) [4 pt] For which $x \in \mathbb{R}$ is f differentiable? Calculate $f'(x)$ for these values of x.

(c) [2 pt] Let A be the set of all roots of the function f (i.e the solutions of $f(x) = 0$). Describe A.

7. [4 pt] Use mathematical induction to prove the statement

$$\frac{d}{dx}x^n = nx^{n-1}$$

for all $n \in \mathbb{N}, n \geq 1$

[Hint: use the product rule]

Total: 36 points