Calculus 1 for AM and TN

Exam October 29, 2021 (13:45 – 15:45)

First Name: ______________________ Last Name: ______________________

Student ID: ____________________________

All answers must be motivated and clearly formulated. Please write your solutions on the provided empty space. The last page of this test is empty. You can also use it if more space is needed. Calculators are not allowed. This exam paper has 5 questions each of which has two parts.

Question:	1	2	3	4	5	Total
Points:	7	6	8	8	7	36
Score:						

Question 1. (7 points)

(a) (3 points) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be mutually perpendicular vectors. Determine the projection of vector $2\mathbf{u} + 3\mathbf{v} + 4\mathbf{w}$ onto $\mathbf{v}$.

(b) (4 points) Determine an equation of a plane that contains the points $P = (-2, -2, 0)$, $Q = (-1, -3, -1)$ and $R = (0, -1, -1)$.

Question 2. (6 points)

(a) (4 points) Find all the complex solutions of the equation $z^2 - \sqrt{3}z + 1 = 0$. Moreover, write these complex solutions in Euler form.

(b) (2 points) Let $z_1 = |z_1|e^{\theta_1 i}$, $z_2 = |z_2|e^{\theta_2 i}$ be two complex numbers, such that $0 \leq \theta_1 \leq \frac{\pi}{2}$ and $0 \leq \theta_2 \leq \frac{\pi}{2}$. For which values of θ_1 and θ_2 is the product $z_1 \cdot z_2$ a real number?

Question 3. (8 points)

(a) (4 points) Find the solutions of the differential equation

$$y'' - 6y' + 9y = e^{3x} + 3.$$

(b) (4 points) Solve the first order differential equation

$$(x+2)y' + y = (x+2)^{11} \quad \text{with} \quad x > -2.$$

Question 4. (8 points)

(a) (4 points) Given that

$$f(x) = \begin{cases} x^3 + 2x - 4 & \text{if } 0 \leq x \leq 1 \\ x^2 + c & \text{if } 1 < x \leq 2, \end{cases}$$

determine the constant c so that the Intermediate Value Theorem is applicable to f. Using the Intermediate Value Theorem show that the function $f(x)$ has a zero in the interval $(0, 2)$.

(b) (4 points) Consider the function $f(x) = x^3 + x$. Let x_0 be a real number. By using the definition of derivative, determine $f^{'}(x_0)$, the derivative of $f(x)$ at x_0.

Question 5. (7 points)

(a) (5 points) Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be two continuous functions with derivative $f^{'}(x) = g^{'}(x)$ for all $x \in (a, b)$.

State the Mean Value Theorem for the function $(f - g)(x)$.

Use the Mean Value Theorem to show that $f(x) = g(x) + C$ for some constant $C \in \mathbb{R}$ for all $x \in [a, b]$.

Note: The function $(f - g)(x)$ is defined as $(f - g)(x) = f(x) - g(x)$ for all x.

(b) (2 points) Evaluate the limit

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right).$$