1 (a) Let u, v, w be mutually perpendicular vectors.

Projection of vector $2u+3v+4w$ onto v is

$$\text{Proj}_v(2u+3v+4w) = \left(\frac{(2u+3v+4w)\cdot v}{|v|^2}\right) v$$

$$= \left(\frac{2u\cdot v + 3\, v\cdot v + 4\, w\cdot v}{|v|^2}\right) v$$

$$= \left(\frac{0 + 3\,|v|^2 + 0}{|v|^2}\right) v = \left(\frac{3\cancel{|v|^2}}{\cancel{|v|^2}}\right) v$$

$$= 3v$$

1(b) An equ. of a plane that contains points $P(-2,-2,0)$, $Q(-1,-3,-1)$ and $R(0,-1,-1)$.

$\overrightarrow{PQ} = \langle 1, -1, -1 \rangle$.

$\overrightarrow{PR} = \langle 2, 1, -1 \rangle$

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= i \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$= 2i - j + 3k$

$\overrightarrow{PX} = \langle x+2, y+2, z \rangle$

The plane equation is $\vec{n} \cdot \overrightarrow{PX} = 0$.

Hence, $\langle 2, -1, 3 \rangle \cdot \langle x+2, y+2, z \rangle = 0$

$$2(x+2) + (-1).(y+2) + 3.z = 0$$

$$2x + 4 - y - 2 + 3z = 0$$

$$2x - y + 3z + 2 = 0$$

$$\boxed{2x - y + 3z = -2}$$

(2) (a_1) $z^2 - \sqrt{3}\,z + 1 = 0$

$$z_1, z_2 = \frac{\sqrt{3} \mp \sqrt{3 - 4.1.1}}{2.1} = \frac{\sqrt{3} \mp \sqrt{-1}}{2}$$

$$\boxed{z_1 = \frac{\sqrt{3}}{2} + i.\frac{1}{2}} \quad \text{and} \quad \boxed{z_2 = \frac{\sqrt{3}}{2} - i.\frac{1}{2}}$$

(a_2) $|z_1| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{4}{4}} = \boxed{1}$

$z_1 = 1.\, e^{i\theta_1}$ Where $\cos(\theta_1) = \frac{\sqrt{3}}{2}$ and $\sin(\theta_1) = \frac{1}{2}$.

Hence $\theta_1 = \frac{\pi}{6}$.

$z_1 = e^{i\pi/6}$.

$z_2 = e^{i(2\pi - \pi/6)} = e^{i\frac{11\pi}{6}}$ or $z_2 = e^{i(-\pi/6)}$.

(b) $z_1 . z_2 = |z_1|.|z_2|.\, e^{(\theta_1+\theta_2)i}$

where

$e^{(\theta_1+\theta_2)i} = \cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2)$

$z_1 . z_2 \in \mathbb{R} \iff \sin(\theta_1+\theta_2) = 0$

$\Rightarrow \boxed{\theta_1+\theta_2 = 0}$ or $\boxed{\theta_1+\theta_2 = \pi}$.

since $0 \leq \theta_1 \leq \frac{\pi}{2}$ and $0 \leq \theta_2 \leq \frac{\pi}{2}$

$\theta_1 + \theta_2 = 0 \Rightarrow \theta_1 = \theta_2 = 0$.

and

$\theta_1 + \theta_2 = \pi \Rightarrow \theta_1 = \theta_2 = \frac{\pi}{2}$.

3(a) Find the solutions of the DE $y'' - 6y' + 9y = e^{3x} + 3$.

$$r^2 - 6r + 9 = 0 \Rightarrow r_1, r_2 = \frac{6 \mp \sqrt{36-36}}{2.1} = 3.$$

$$y_1 = e^{3x} \quad \text{and} \quad y_2 = x.e^{3x}$$

$$y_h = \alpha e^{3x} + \beta x. e^{3x}$$

$$y_p = A x^2 e^{3x} + B$$

$$y_p' = A[2x.e^{3x} + x^2. 3e^{3x}]$$

$$y_p'' = A e^{3x}[2 + 12x + 9x^2]$$

As we substritute these y_p, y_p' and y_p'' we get

$$Ae^{3x}[2+12x+9x^2] - 6Ae^{3x}[2x+3x^2] + 9[Ax^2e^{3x}+B] = e^{3x}+3$$

This equation simplifies to

$$2Ae^{3x} + 9B = e^{3x} + 3 \Rightarrow 2A = 1 \quad \text{and} \quad 9B = 3$$

$$\Rightarrow A = \frac{1}{2} \quad \text{and} \quad B = \frac{1}{3}$$

$$y = \alpha e^{3x} + \beta x.e^{3x} + \frac{1}{2}x^2. e^{3x} + \frac{1}{3}.$$

3(b) $(x+2)\,y' + y = (x+2)^{11}$ with $x > -2$

$$y' + \frac{1}{x+2} \cdot y = (x+2)^{10}$$

$$v(x) = e^{\int \frac{1}{x+2}\,dx} = e^{\ln(x+2)} = (x+2).$$

$$(x+2) \cdot y' + y = (x+2)^{11}.$$

$$\big((x+2) \cdot y\big)' = (x+2)^{11}.$$

$$(x+2) \cdot y = \int (x+2)^{11}\,dx = \frac{(x+2)^{12}}{12} + C.$$

$$\Rightarrow \boxed{y(x) = \frac{(x+2)^{11}}{12} + \frac{C}{(x+2)}}$$

4) a) We check the continuity of $f(x)$ on $[0,2]$ interval in order to apply the Intermediate Value Theorem.

$$f(x) = \begin{cases} x^3 + 2x - 4 & \text{if } 0 \leq x \leq 1 \\ x^2 + c & \text{if } 1 \leq x \leq 2 \end{cases}$$

is cts on $(0,1)$ since x^3+2x-4 is a polynomial and similarly $f(x)$ is cts. on $(1,2)$ since x^2+c is a polynomial.

We now need to check the continuity of $f(x)$ at $x=1$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^3+2x-4 = -1 = f(1)$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2+c = 1+c$$

The continuity at $x=1$ holds if $-1 = 1+c$. That is $c = -2$.

Since $f(0) = -4$ and $f(2) = 2^2+(-2) = 2$, by Intermediate Value theorem we know that for all s between $f(0)=-4$ and $f(2)=2$ there exists a $c \in [0,2]$ such that $f(c)=s$.

0 is between $f(0)=-4$ and $f(2)=2$. Therefore, there exists a $c \in [0,2]$ s.t. $f(c)=0$.

4 b) Let $f(x) = x^3 + x$. Let $x_0 \in \mathbb{R}$. Find $f'(x_0)$.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{\left((x_0+h)^3 + (x_0+h)\right) - \left(x_0^3 + x_0\right)}{h}$$

$$= \lim_{h \to 0} \frac{(x_0+h)^3 - x_0^3 + h}{h}$$

$$= \lim_{h \to 0} \frac{\left((x_0+h) - x_0\right)\left((x_0+h)^2 + (x_0+h).x_0 + x_0^2\right) + h}{h}$$

$$= \lim_{h \to 0} \frac{h\left((x_0+h)^2 + (x_0+h).x_0 + x_0^2\right) + h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}.\left((x_0+h)^2 + (x_0+h).x_0 + x_0^2 + 1\right)}{\cancel{h}}$$

$$= \lim_{h \to 0} (x_0+h)^2 + (x_0+h).x_0 + x_0^2 + 1$$

$$= x_0^2 + x_0^2 + x_0^2 + 1 = 3x_0^2 + 1.$$

Mean Value Theorem (MVT)

5 (a) Let $f-g : [a,b] \to \mathbb{R}$,

$f-g$ is continuous on $[a,b]$ and $f-g$ is differentiable on (a,b). Then $\underbrace{\exists}_{\text{there exists}}$ $c \in (a,b)$ such that

$$(f-g)'(c) = \frac{(f-g)(b) - (f-g)(a)}{b-a}.$$

We want to show, using MVT, that $f(x) = g(x) + C$, for all $x \in [a,b]$. Equivalently, we want to show that $(f-g)(x) = C$ for all $x \in [a,b]$.

Let $x, y \in [a,b]$ such that $x < y$.

Since $[x,y] \subseteq [a,b]$ $(f-g)$ is cts. on $[x,y]$ and differentiable on (x,y).

By the MVT there exists $c \in (x,y)$ such that

$$(f-g)'(c) = \frac{(f-g)(y) - (f-g)(x)}{y-x}.$$

Since $f'(x) = g'(x)$ for all $x \in (a,b)$

$\Rightarrow$ $(f-g)'(x) = 0$ for all $x \in (a,b)$.

Therefore, $(f-g)'(c) = 0$.

$$\frac{(f-g)(y) - (f-g)(x)}{y-x} = 0 \Rightarrow (f-g)(y) = (f-g)(x)$$ where $x, y \in [a,b]$

This implies that $(f-g)(x)$ is a constant function. That is, $(f-g)(x) = C$ for some constant C.

$(f-g)(x) = f(x) - g(x) = C \Rightarrow f(x) = g(x) + C$

5 (b) $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right)$

$$-1 \leqslant \cos\left(\frac{1}{x^2}\right) \leqslant 1.$$

Since $x^2 > 0$ as $x \to 0$.

$$-x^2 \leqslant x^2 \cdot \cos\left(\frac{1}{x^2}\right) \leqslant x^2 \cdot 1$$

$\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0.$

By Sandwich thrm.

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0.$$