Math β 2 13-01-2017

$$f(x) = \begin{cases} \dfrac{x^2}{3+\sin(1/x)} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(a) $x \neq 0$ $f'(x) = \dfrac{2x(3+\sin 1/x) - \cos(\frac{1}{x}) \cdot \frac{-1}{x^2} \cdot x^2}{(3+\sin(\frac{1}{x}))^2}$

$$= \frac{2x}{3+\sin(\frac{1}{x})} + \frac{\cos\frac{1}{x}}{(3+\sin\frac{1}{x})^2}$$

(b) $f'(0) = \lim_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim_{h\to 0} \dfrac{1}{h} \dfrac{h^2}{3+\sin\frac{1}{h}} = \lim_{h\to 0} \dfrac{h}{3+\sin\frac{1}{h}} = 0$

because $\dfrac{h}{4} \leq \dfrac{h}{3+\sin\frac{1}{h}} \leq \dfrac{h}{2}$

en $\frac{h}{4}, \frac{h}{2} \to 0$ $(h \to 0)$.

(c) $f'(\frac{1}{2k\pi}) = \dfrac{\frac{1}{k\pi}}{3} + \dfrac{1}{3^2} \left(\to \frac{1}{3^2} \; \cancel{(k\to\infty)} (k \to \infty).\right)$

(d) $\lim_{k\to\infty} f'(\frac{1}{2k\pi}) = \frac{1}{9} \neq f'(0) \Rightarrow f'(x)$ not continuous in 0.

2\. a. $f: [a,b]$ continuous and differentiable on (a,b).
There exists $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

b. $f'(x) = 0$ $\forall x \in (a,b)$. $\exists c \in (c_1, c_2)$ s.t. $f'(c) = \dfrac{f(c_2)-f(c_1)}{c_2 - c_1}$

$\Rightarrow f(c_2) - f(c_1) = 0 \Rightarrow f(x)$ is constant.

3. $$\lim_{(x,y)\to(0,0)} \frac{x\tan y}{x^2+y^2} \ ?$$

Along $y=0$: $$\lim_{(x,0)\to(0,0)} \frac{x\tan y}{x^2+y^2} = 0.$$

Along $x=y$: $$\lim_{x\to 0} \frac{x\cdot\tan x}{2x^2} = \lim_{x\to 0} \frac{x}{x}\cdot\frac{\tan x}{x}$$

$$= \lim_{x\to 0}\frac{x}{x}\cdot \lim_{x\to 0}\frac{1}{\cos x}\frac{\sin x}{x} = 1.1 = 1 \neq 0$$

limit does not exist.

4. $$S_n = \sum_{k=1}^{n} \frac{2k-1}{n^2}\cdot\left(\frac{k}{n}\right)^{-1} \qquad \Delta x_k = \frac{2k-1}{n^2}$$

(a) $n=6$: $\Delta x_1 = \frac{1}{36}$ $\Delta x_2 = \frac{3}{36}$ $\Delta x_3 = \frac{5}{36}$ $\Delta x_4 = \frac{7}{36}$ $\Delta x_4 = \frac{9}{36}$

$\Delta x_6 = \frac{11}{36}$

$x_1 = \frac{1}{36}$ $x_2 = \frac{4}{36}$ $x_3 = \frac{9}{36}$ $x_4 = \frac{16}{36}$ $x_5 = \frac{25}{36}$ $x_6 = \frac{36}{36} = 1.$

(b) $x_k = \frac{k^2}{n^2}$ $k = 0, \to n.$ $P_n = \left\{0, \frac{1}{n^2}, \frac{2^2}{n^2}, \to \frac{(n-1)^2}{n^2}, 1\right\}.$

(c) $$S_n = \sum_{k=1}^{n} \frac{2k-1}{n^2}\cdot f\left(\frac{k^2}{n^2}\right) \qquad \Rightarrow f\left(\frac{k^2}{n^2}\right) = \frac{n}{k}$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{x}}$$

$$\Rightarrow \lim_{n\to\infty} S_n = \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_0^1 = 2.$$

5. $\bar{f} = \frac{1}{\sqrt[3]{2}-1} \int_1^{\sqrt[3]{2}} x^8 \ln(x^3)\, dx$

$$= \left(\frac{1}{9} x^9 \ln(x^3) \Big|_1^{\sqrt[3]{2}} - \frac{1}{9} \int_1^{\sqrt[3]{2}} x^9 \frac{1}{x^3} \cdot 3x^2\, dx \right) \Big/ (\sqrt[3]{2}-1)$$

$$= \frac{1}{9} 8 . \ln 2 - \frac{1}{3} \int_1^{\sqrt[3]{2}} x^8\, dx \Big/ \ldots$$

$$= \frac{8}{9} . \ln 2 - \frac{1}{27} x^9 \Big|_1^{\sqrt[3]{2}} = \frac{8}{9} \ln 2 - \frac{8}{27} + \frac{1}{27} \Big/ \ldots$$

$$= \left(\frac{8}{9} \ln 2 - \frac{7}{27} \right) \frac{1}{\sqrt[3]{2}-1}$$

alternatively

$$\int_1^{\sqrt[3]{2}} x^8 \ln x^3\, dx \underset{\substack{y = x^3 \\ dy = 3x^2 dx}}{=} \frac{1}{3} \int_1^2 y^2 \ln y\, dy$$

$$= \frac{1}{3} \frac{1}{3} y^3 . \ln y \Big|_1^2 - \frac{1}{9} \int_1^2 y^3 . \frac{1}{y}\, dy$$

$$= \frac{8}{9} \ln 2 - \frac{1}{27} y^3 \Big|_1^2 = \frac{8}{9} \ln 2 - \frac{8}{27} + \frac{1}{27}$$

6. $f(x) = xe^{x^2}$

(a) $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$

$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \dots$

$xe^{x^2} = \underbrace{x + x^3 + \frac{1}{2}x^5}_{} + \frac{1}{6}x^7 \dots$

(b) $xe^{x^2} = x \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$

convergence of $\sum_{k=0}^{\infty} \frac{y^k}{k!}$ on $\mathbb{R}$.

hence also $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!}$.