1.

(a) $S_n = \sum_{k=1}^{n} \frac{k^2+kn}{n^3}$

$= \frac{1}{n}\sum_{k=1}^{n} \frac{k^2+kn}{n^2}$

$= \frac{1}{n}\sum_{k=1}^{n} \underbrace{\left(\frac{k}{n}\right)^2+\left(\frac{k}{n}\right)}_{f\left(\frac{k}{n}\right)} \Rightarrow f(x)=x^2+x.$

(b) $\lim_{n\to\infty} S_n = \int_0^1 f(x)\,dx = \int_0^1 x^2+x\,dx$

$= \frac{1}{3}x^3+\frac{1}{2}x^2\Big|_0^1 = \frac{1}{3}+\frac{1}{2} = 5/6$

Alternative calculation:

$\sum_{k=1}^{n} k = \cancel{\tfrac{1}{2}k(k+1)}\ \frac{1}{2}n(n+1)$

$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(1+2n)$

hence $S_n = \frac{1}{n^3}\sum_{k=1} k^2 + \frac{1}{n^2}\sum_{k=1} k$

$= \underbrace{\underbrace{\frac{1/6\, n(n+1)(2n+1)}{n^3}}_{\downarrow\ 1/3} + \underbrace{\frac{\frac{1}{2}n(n+1)}{n^2}}_{\downarrow\ 1/2}}_{5/6}$

2. (a) $f : [a,b] \to \mathbb{R}$ continuous, then there exists $c \in [a,b]$, such that $\int_a^b f(x)\,dx = f(c).(b-a)$

(b) (AM, AT, EE)

$$\int_0^{\sqrt{\pi}} x^3 \sin x^2\,dx \underset{\substack{y=x^2 \\ dy=2x\,dx}}{=} \frac{1}{2}\int_0^{\pi} y \sin y\,dy$$

$$= -\frac{1}{2}\, y.\cos y \Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos y\,dy$$

$$= +\frac{1}{2}\pi + \frac{1}{2}\sin y \Big|_0^{\pi} = \frac{1}{2}\pi$$

$$\Rightarrow \bar{f} = \frac{1}{2}\pi \Big/ \sqrt{\pi} = \frac{1}{2}\sqrt{\pi}$$

(b') TN

$$\iint_{\frac{\pi}{4} \le x^2+y^2 \le \frac{\pi}{2}} \sin(x^2+y^2)\,dA$$

$x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$

$\sqrt{\frac{\pi}{4}} \le r \le \sqrt{\frac{\pi}{2}}$
$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \int_{\sqrt{\pi/4}}^{\sqrt{\pi/2}} r \sin r^2\,dr\,d\theta$$

$$= \cancel{2\pi \left(\frac{1}{2}\sin r^2 \Big|_{\sqrt{\pi/4}}^{\sqrt{\pi/2}}\right) = \pi\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right)}$$

$$= 2\pi \left(-\frac{1}{2}\cos r^2 \Big|_{\sqrt{\pi/4}}^{\sqrt{\pi/2}}\right) = -\pi\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right)$$

$$= \pi.\frac{1}{2}\sqrt{2}$$

Area of strip:

$$\pi\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\pi^2/4$$

$\Rightarrow$ average value:

$$\bar{f}=\frac{4}{\pi^2}\cdot\pi\cdot\frac{1}{2}\sqrt{2}=\frac{2\sqrt{2}}{\pi}$$

3. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{(n+1)!}$ $\qquad a_n=\frac{n^3}{(n+1)!}$

ratio test $\displaystyle\frac{a_{n+1}}{a_n}=\frac{(n+1)^3}{(n+2)!}\cdot\frac{(n+1)!}{n^3}$

$$=\left(\frac{n+1}{n}\right)^3\cdot\frac{1}{n+2}\to 0$$

$\Rightarrow$ convergent.

4. $x(t)=t^2 \quad y(t)=t^3 \quad \dot{x}(t)=2t \quad \dot{y}(t)=3t^2$

$$|\gamma|=\int_0^2\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}\,dt=\int_0^2\sqrt{4t^2+9t^4}\,dt$$

$$=\int_0^2 t\cdot\sqrt{4+9t^2}\,dt \underset{\substack{s=t^2\\ ds=2t\,dt}}{=} \frac{1}{2}\int_0^4\sqrt{4+9s}\,ds$$

$$=\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{9}(4+9s)^{3/2}\Big|_0^4$$

$$=\frac{1}{27}\left(40^{3/2}-4^{3/2}\right)=\frac{1}{27}\left(40\sqrt{40}-4\sqrt{4}\right)$$

$$=\frac{1}{27}\left(80\sqrt{10}-8\right).$$

5. $f(x,y) = x^4 - 6x^2y^2 + y^4$

(a) $\frac{\partial f}{\partial x}(x,y) = 4x^3 - 12xy^2$

$\frac{\partial f}{\partial y}(x,y) = 4y^3 - 12yx^2$

(b) grad $f(x,y) = 0 \iff$

$4x^3 - 12xy^2 = 4x(x^2 - 3y^2) = 0$

and $4y(y^2 - 3x^2) = 0$

if $x \neq 0$, then $x^2 = 3y^2 \Rightarrow y \neq 0$

$\Rightarrow y^2 = 3x^2 \Rightarrow x^2 = 3y^2 = 3.3x^2 = 9x^2$

$\Rightarrow x = 0$ ↯

$\Rightarrow x = 0 \wedge y = 0$

only one critical point: $(0,0)$

(c) $f_{xx} = 12x^2 - 12y^2$ $\qquad f_{yy} = 12y^2 - 12x^2$

$f_{xy} = -24y$

All zero in $(0,0) \Rightarrow$ no conclusion

However:

$$\left.\begin{array}{l} f(x,0) = x^4 \rightarrow \text{minimum} \\ f(x,x) = -4x^4 \rightarrow \text{maximum} \end{array}\right] \rightarrow \text{saddle point}$$

d. define $g(x,y) = x^2+y^2-1$.

Lagrange equations:

$\circ$ $4x^3-12xy^2+\lambda \cdot 2x = 0$

$\circ\circ$ $4y^3-12yx^2+\lambda \cdot 2y = 0$

$\circ\circ\circ$ $x^2+y^2=1$

$\circ$ $x(4x^2-12y^2+2\lambda) = 0$

$\circ\circ$ $y(4y^2-12x^2+2\lambda) = 0$

$x=0 \Rightarrow y=\pm 1 \Rightarrow 4+2\lambda=0 \Rightarrow \lambda=-2.$

$y=0 \Rightarrow x=\pm 1 \Rightarrow 4+2\lambda=0 \Rightarrow \lambda=-2.$

$x\neq 0, y\neq 0$

$$\Rightarrow \quad 4x^2-12y^2+2\lambda=0$$
$$4y^2-12x^2+2\lambda=0$$
$$\Rightarrow \quad 4x^2-12y^2=4y^2-12x^2$$
$$\Rightarrow \quad \left.\begin{array}{l} 16x^2=16y^2 \Rightarrow x^2=y^2 \\ x^2+y^2=1 \end{array}\right\} \Rightarrow 2x^2=1$$

$$\Rightarrow \quad x=\pm\tfrac{1}{2}\sqrt{2};\ y=\pm\tfrac{1}{2}\sqrt{2}$$

critical points:

	$(\pm1, 0)$,	$(0,\pm1)$,	$(\pm\frac{1}{2}\sqrt{2},$	$\pm\frac{1}{2}\sqrt{2})$
f:	1	1	-1	-1
	local max	loc max	loc min	loc min

~~loc~~

5f. $f(x,y) = x^4 - 6x^2y^2 + y^4$

$f(1,1) = -4.$

$$x^4 - 6x^2 y(x)^2 + y(x)^4 = -4$$

$$\Rightarrow \quad 4x^3 - 12x\, y(x)^2 - 12x^2 y(x) y'(x)$$
$$+ 4y(x)^3 y'(x) = 0$$

substitute $x=1$, $y(1)=1$:

$$4 - 12 - 12y'(1) + 4y'(1) = 0$$

$$\Rightarrow \quad -8 - 8y'(1) = 0$$

$$\Rightarrow \quad y'(1) = -1.$$