Calculus 2 for Applied Mathematics & Applied Physics

Code	201800136/201800158
Date:	29 January 2021
Time:	09:00-12:00
Type of test	closed book
Allowed aids	nothing

Please provide motivation for all your answers and calculations. The use of electronic devices is not allowed.

1. Let S_n be given by

$$S_n = \sum_{k=1}^{n} \frac{k+n}{\sqrt{kn^3}}.$$

(a) (3p) Interpret S_n as a Riemann sum of a function $f(x)$ on the interval $[0, 1]$. Hint: take the partition $P_n = \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$ as the starting point for rewriting S_n as Riemann sum and determine the function $f(x)$.

(b) (3p) Now calculate

$$\lim_{n\to\infty} S_n.$$

2. The region $D \subset \mathbb{R}^2$ is the set enclosed by the lines

$$\ell_1 : x - y = -1 \quad \ell_2 : x - y = 1 \quad \ell_3 : x = 0 \quad \ell_4 : y = 0 \text{ and the curve } \gamma\text{: xy=1.}$$

Figure 1: Region D

(a) (2p) Denote the point of intersection γ with ℓ_1 by $P_1 = (x_1, y_1)$ and the point of intersection of γ with ℓ_2 by $P_2 = (x_2, y_2)$. Calculate x_1 and x_2.

(b) (2p) Use x_1 and x_2 from Part 2a to split D in (three) parts and provide an expression for

$$\iint_D (x + y)\, \mathrm{d}A$$

with the limits of integration for each part.

(c) (3p) Calculate the integral over the first part:

$$\iint_{D_1} (x + y)\, \mathrm{d}A$$

directly in Cartesian coordinates (x, y).

(d) (4p) Using the transformation $u = x - y$, $v = xy$, determine the limits of integration for u and v, the Jacobian and calculate

$$\iint_D (x + y)\, \mathrm{d}A.$$

3. (5p) Let the planar curve γ be given by

$$\gamma = \{(\cos(t) + t\sin(t), \sin(t) - t\cos(t))\} \quad 0 \leq t \leq 2\pi.$$

Calculate the length of γ.

4. Let $f(x)$ be given by the power series:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2k+1} x^{2k+1}.$$

(a) (3p) Determine the radius of convergence of the power series.

(b) (4p) Determine the function $f(x)$.

5. Let D be given by

$$\{(x,y) \mid x^2+y^2 \leq 1\}$$

and $f(x,y)$ by

$$f(x,y) = x^2 + 10xy + y^2$$

See Figure 2 for an impression of the graph of $f(x,y)$.

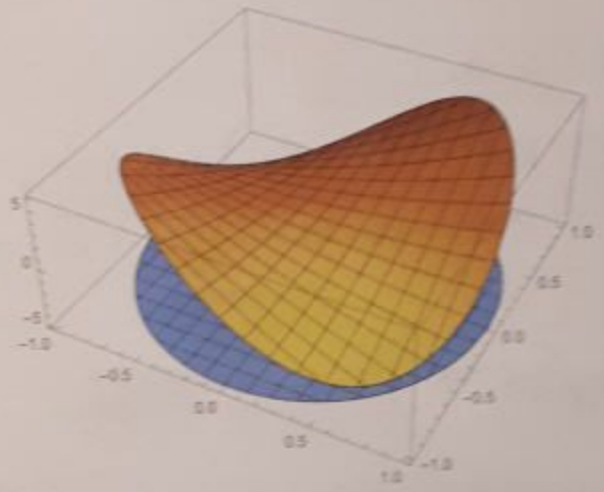

Figure 2: Graph of $f(x,y)$

(a) (2p) Determine the critical points of $f(x,y)$ in the interior of D.

(b) (4p) Show that $f(x,y)$ has a saddle point at the origin.

(c) (2p) Determine an equation of the tangent plane to the graph of $f(x,y)$ at the origin.

(d) (5p) Use the method of Lagrange Multipliers to find the extremal values, including their nature, of $f(x,y)$ on the boundary of D.

(e) (3p) What is the maximal directional derivative of $f(x,y)$ at the point $(\frac{1}{2},\frac{1}{2})$?

6. (a) (1p) (1p) Verify that

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

(b) (4p) Calculate

$$\lim_{x \to \infty} (\sqrt[3]{x^3 + x^2} - x).$$

Hint: with an appropriate choice of a and b, use 6a.