Mathematics B2: Newton, solution to test January 27, 2014

1. Note that expression becomes $\frac{\infty}{\infty}$ to $x \to \infty$ [$\frac{1}{2}$pt]

   proceed by algebra (dividing by x) or analysis (l'Hôpital) [$\frac{1}{2}$pt]

   $$\lim_{x\to\infty} \frac{\sqrt{2x^2 - 6x + 2}}{x + 3} = \lim_{x\to\infty} \frac{\sqrt{2 - \frac{6}{x} + \frac{2}{x^2}}}{1 + \frac{3}{x}}$$ [$1\frac{1}{2}$pt]

   answer $\sqrt{2}$ [$\frac{1}{2}$pt]

   $\Big[$or l'Hospital $\lim_{x\to\infty} \frac{2x - 3}{\sqrt{2x^2 - 6x + 2}}$ does not help [1pt]$\Big]$

2. (a) f is continuous at 0 : $\lim_{x\to 0} f(x) = f(0)$ [1pt]

   (b) $\lim_{x\to 0} x \ln(x^2) = 0 \cdot \infty$ indefinite [$\frac{1}{2}$pt]

   $\lim_{x\to 0} x \ln(x^2) = \lim_{x\to 0} \frac{\ln(x^2)}{\frac{1}{x}}$ for applying l'Hôpital [$\frac{1}{2}$pt]

   $$\lim_{x\to 0} \frac{\ln(x^2)}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x^2} \cdot 2x}{-\frac{1}{x^2}} = \lim_{x\to 0} -2x$$ [$\frac{1}{2}$pt]

   answer: 0 [$\frac{1}{2}$pt]

   (c) endpoints $f(-1) = 0$ and $f(1) = 0$ [$\frac{1}{2}$pt]

   evaluating at at (possible) critical point $f(0) = 0$ [$\frac{1}{2}$pt]

   for other critical points $f'(x) = \ln(x^2) + 2 = 0$ [$\frac{1}{2}$pt]

   $\ln(x^2) + 2 = 0 \Leftrightarrow x^2 = e^{-2} \Leftrightarrow x = e^{-1} or x = -e^{-1}$ [1pt]

   $f(e^{-1}) = -\frac{2}{e}$ and $f(-e^{-1}) = \frac{2}{e}$ [$\frac{1}{2}$pt]

   conclusion:

   absolute max $f(-e^{-1}) = \frac{2}{e}$ [$\frac{1}{2}$pt]

   absolute min $f(e^{-1}) = -\frac{2}{e}$ [$\frac{1}{2}$pt]

3. (a) f is continuous if $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$ [$\frac{1}{2}$pt]

or $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{y^5}{y^4} = \lim_{y\to 0} y = 0$

$\lim_{x\to 0} f(x,0) \lim_{x\to 0} \frac{2x^2}{x^2} = 2$ [1pt]

Along x-axis approaching $(0,0)$ gives value $2 \neq 0$ [1pt]

Conclusion: f is not continuous at $(0,0)$ [$\frac{1}{2}$pt]

(b) $\frac{\partial f}{\partial x} = \frac{4x(x^2+y^4) - 2x(2x^2+y^5)}{(x^2+y^4)^2} = \frac{4xy^4 - 2xy^5}{(x^2+y^4)^2}$ [$1\frac{1}{2}$pt]

$\frac{\partial f}{\partial y} = \frac{5y^4(x^2+y^4) - 4y^3(2x^2+y^5)}{(x^2+y^4)^2} = \frac{5x^2y^4 + y^8 - 8x^2y^3}{(x^2+y^4)^2}$ [$1\frac{1}{2}$pt]

(c) $f(0,1) = 1$ [$\frac{1}{2}$pt]

$\frac{\partial f}{\partial x}(0,1) = 0$ and $\frac{\partial f}{\partial y}(0,1) = 1$ [1pt]

tangent plane: $z - 1 = 0(x-0) + 1(y-1)$ [$\frac{1}{2}$pt]

4. Correct use product rule for product $x \cdot \int_1^x \frac{t}{1+t^4} dt$ [1pt]

correct use fundamental theorem for $\frac{d}{dx} \int_1^x \frac{t}{1+t^4} dt$ [1pt]

Answer: $1 \cdot \int_1^x \frac{t}{1+t^4} dt + x \cdot \frac{x}{1+x^4}$ [1pt]

5. (a) Find improperty $\int_0^\infty \frac{x}{(1+x)^2}dx = \lim_{t\to\infty}\int_0^t \frac{xdx}{(1+x^2)^2}$ [$\frac{1}{2}$pt]

antiderivative of $\frac{x}{(1+x^2)^2}$ by substitution $u = x^2$ [1pt]

$$\int \frac{x}{(1+x^2)^2}dx = \int \frac{\frac{1}{2}du}{(1+u)^2} = \frac{-\frac{1}{2}}{1+u} = \frac{-\frac{1}{2}}{1+x^2}$$ [1pt]

$$\int_0^t \frac{x}{(1+x^2)^2}dx = \frac{-\frac{1}{2}}{1+t^2} - -\frac{1}{2} = \frac{1}{2} - \frac{\frac{1}{2}}{1+t^2}$$ [$\frac{1}{2}$pt]

$$\int_0^\infty \frac{x}{(1+x^2)^2}dx = \lim_{t\to\infty}\left(\frac{1}{2} - \frac{\frac{1}{2}}{1+t^2}\right) = \frac{1}{2}$$ [1pt]

(b) first partial integration $\int x\ln^2(x)dx = \int \ln^2(x)d(\frac{1}{2}x^2)$

$$= \frac{1}{2}x^2\ln^2(x) - \int \frac{1}{2}x^2 \cdot 2\ln(x)\frac{1}{x}dx$$ [$1\frac{1}{2}$pt]

second partial integration $\int x\ln(x)dx = \int \ln(x)d(\frac{1}{2}x^2)$

$$= \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x}dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2$$ [$1\frac{1}{2}$pt]

answer: $\frac{1}{2}x^2\ln^2(x) - \frac{1}{2}x^2\ln(x) + \frac{1}{4}x^2 + C$ [1pt]

6. $\sum_{n=0}^\infty \left(x^2 - \frac{1}{2}\right)^n$ is basic geometric series in $x^2 - \frac{1}{2}$ [$\frac{1}{2}$pt]

convergent in case $-1 < x^2 - \frac{1}{2} < 1$ [$\frac{1}{2}$pt]

$-1 < x^2 - \frac{1}{2} < 1 \Leftrightarrow -\sqrt{1\frac{1}{2}} < x < \sqrt{1\frac{1}{2}}$(interval) [1pt]

$$\sum_{n=0}^\infty \left(x^2 - \frac{1}{2}\right)^n = \frac{\text{first term}}{1 - \text{ratio}} = \frac{1}{1-(x^2-\frac{1}{2})} = \frac{1}{1\frac{1}{2} - x^2}$$ [1pt]

7. Taylor series $\sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!}x^k$ [1pt]

$f(x) = \cos(2x); f'(x) = -2\sin(2x); f''(x) = -4\cos(2x); f'''(x) = 8\sin(2x)$

$f(0) = 1; f'(0) = 0; f''(0) = -4; f'''(0) = 0$ [2pt]

answer: $1 - \frac{4}{2}x^2 + \frac{16}{24}x^4 - \frac{64}{720}x^6 + ...$ [1pt]

or $\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + ...$ [2pt]

substitute $t = 2x$ [1pt]

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + ...$$ [1pt]