Exam Stochastic Differential Equations (Mastermath SDE)
08-06-2015; 13:30 – 16:30.

No calculator/phone allowed

(5) 1. a. Let $X \in L^1$ be a random variable and let $\mathcal{F}$ and $\mathcal{G}$ be σ-algebras such that $\mathcal{F} \subseteq \mathcal{G}$. Show that $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{F}) = \mathbb{E}(X|\mathcal{F})$.

(5) b. Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of random variables. Assume there is a constant C such that for all integers $n \in \mathbb{N}$, $\mathbb{E}(|X_n|^2) \leq C$. Show that X is uniformly integrable.

(5) c. Let $(X_t)_{t\geq 0}$ be a nonnegative local martingale such that $\mathbb{E}(X_0) < \infty$. Show that X is a supermartingale.

(2) d. Let $0 < a < b$ and assume $\xi : \Omega \to \mathbb{R}$ is $\mathcal{F}_a$-measurable. Use the definition to show that $\mathbf{1}_{(a,b]}\xi$ is predictable.

(4) e. Use the previous exercise and an approximation argument to show that any left-continuous adapted process $Z : \mathbb{R} \times \Omega \to \mathbb{R}$ is predictable.

2. Let B be a standard Brownian motion and let $\mathcal{F}_t = \sigma(B_s : s \leq t)$.

(6) a. Let $\alpha \geq 0$. Using the properties of conditional expectations and the independent increments of Brownian motion show that $X_t = \cosh(\alpha|B_t|)\exp(-\alpha^2 t/2)$ is a martingale.

Hint: Recall that $\cosh(|x|) = \cosh(x) = (e^x + e^{-x})/2$. You may also use the identity: $\mathbb{E}(\exp(\xi)) = \exp(\sigma^2/2)$ for $\xi \sim N(0, \sigma^2)$.

Fix $A > 0$ and let $\tau = \inf\{t \geq 0 : |B_t| = A\}$.

(3) b. Prove that τ is a stopping time.

(3) c. Show that $\mathbb{E}(X_{t\wedge\tau}) = 1$ for all $t \geq 0$.

(5) d. Show that $\mathbb{E}(X_\tau) = 1$ and use this to find a formula for $\mathbb{E}(e^{-\lambda\tau})$ where $\lambda \geq 0$.

Hint: You may use the known fact that $\tau < \infty$ almost surely.

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3. Let M be a continuous L^2-martingale. We will show below, in steps (a) through (c), that $M^2 - [M]$ is a martingale. Let $t > s \geq 0$. Consider a partition of $[0,t]$ given by

$$0 = t_0 < t_1 < \cdots < t_k = s < \cdots < t_n = t.$$

(4) a. Show that for $t_i \geq s$,

$$\mathbb{E}\left[M_{t_{i+1}}^2 - M_{t_i}^2 - (M_{t_{i+1}} - M_{t_i})^2 \middle| \mathcal{F}_s \right] = 0.$$

(2) b. Derive that

$$\mathbb{E}\left[M_t^2 - \sum_{i=0}^{n-1} (M_{t_{i+1}} - M_{t_i})^2 \middle| \mathcal{F}_s \right] = M_s^2 - \sum_{i=0}^{k-1} (M_{t_{i+1}} - M_{t_i})^2.$$

(3) c. Using the previous identity and the existence theorem for quadratic variations of continuous L^2-martingales show that $M^2 - [M]$ is a martingale.

(6) d. Let M and N be continuous L^2-martingales with quadratic covariation $[M,N]_t = 2t - 1$. Let $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be given by $X_t = M_t + \cos(t)$. Let $U : \mathbb{R}_+ \to \mathbb{R}$ and $V : \mathbb{R}_+ \to \mathbb{R}$ be given by $U_t = t$ and $V_t = t^2$. Use the properties of quadratic (co)variations to calculate the $[U \cdot X, V \cdot N]_t$ for $t \geq 0$.

4. Let (B_t) be a standard Brownian motion defined on a (filtered) probability space $(\Omega, \mathcal{F}\ (\mathcal{F}_t), P)$. Suppose X_t is a process satisfying the SDE

$$dX_t = X_t dB_t, \quad X_0 = 1. \qquad (*)$$

Define the process $(Z_t)_{t\in[0,1]}$ by

$$Z_t = X_t e^{-\int_0^t B_s^2 ds}.$$

(6) a. Apply Itô's formula to show that

$$dZ_t = Z_t(dB_t - B_t^2 dt), \quad Z_0 = 1. \qquad (**)$$

(5) b. Find the solution X_t satisfying the SDE $(*)$.

Hint: You may propose a solution and appeal to the uniqueness theorem.

(3) c. Use the solution in (b) to show that $\mathbb{E}(Z_t^2) \leq e^t$.

Hint: You may use the identity $\mathbb{E}(\exp(\xi)) = \exp(\sigma^2/2)$ for $\xi \sim N(0, \sigma^2)$.

(6) d. In the "integrated version" of the SDE $(**)$ two relevant random variables are $\int_0^1 Z_t dB_t$ and $\int_0^1 Z_t B_t^2 dt$. Use (c) to show that both of the following hold.

$$(i)\ \mathbb{E}\left[\left(\int_0^1 Z_t dB_t \right)^2 \right] < \infty \quad \text{and} \quad (ii)\ \mathbb{E}\left[\int_0^1 |Z_t B_t^2| dt \right] < \infty.$$

Hint: If needed, you may assume the expressions for the moments of Gaussian distribution, without proof. For example, $\mathbb{E}(\xi^4) = 3\sigma^4$ for $\xi \sim N(0, \sigma^2)$.

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5. Let $0 < T < \infty$. Suppose H is a predictable process satisfying

$$\int_0^T H_t^2\,dt < \infty, \quad \text{a.s.} \qquad (\dagger)$$

and let $Z_t \equiv Z_t(H)$ be given by

$$Z_t(H) = \exp\left\{\int_0^t H_s\,dB_s - \frac{1}{2}\int_0^t H_s^2\,ds\right\}.$$

(5) a. Use Itô's formula to Z to show that, under $(\dagger)$, Z is a continuous local martingale.

In the Girsanov Theorem an important assumption is that $(Z_t)_{t\in[0,T]}$ is a martingale. Since Z is nonnegative using Question (1c) one sees that Z is a supermartingale. It can be shown that $(Z_t)_{t\in[0,T]}$ is a martingale if and only if $\mathbb{E}[Z_T] = 1$. In this exercise we will check the latter in several steps under the assumption that there is a constant C such that

$$\int_0^T H_t^2\,dt \;\le\; C, \quad \text{a.s.} \qquad (\dagger\dagger)$$

Let $\sigma_n = \inf\{0 \le t \le T : \; Z_t \ge n\}$, where we set $\sigma_n = T$ if the set is empty. In this way $(\sigma_n)_{n\ge1}$ is a localizing sequence for Z. Denote by $Z_t^{(n)}$ the stopped process $Z_{t\wedge\sigma_n}$. It then follows immediately that

$$\mathbb{E}\big(Z_t^{(n)}\big) = 1, \quad n \ge 1,\; 0 \le t \le T.$$

(3) b. Show from the definitions that $Z_t^{(n)} = Z_t(H^{(n)})$ a.s., where $H^{(n)}$ is given by

$$H_s^{(n)}(\omega) = H_s(\omega)\mathbf{1}_{[\,0,\,\sigma_n(\omega)]}(s), \qquad \text{for } 0 \le s \le T.$$

(5) c. Show that if $(\dagger\dagger)$ holds then $\mathbb{E}\Big(\big(Z_T^{(n)}\big)^2\Big) \le e^C$, for all $n \ge 1$.

Hint: Argue that $\left(Z_t^{(n)}\right)^2 \le M_t^{(n)}\,e^C$ for some suitable martingale $M_t^{(n)}$ with $M_0^{(n)} = 1$.

(4) d. Show that, under $(\dagger\dagger)$, $\mathbb{E}[Z_T] = 1$.

Hint: Argue that $\sigma_n \to T$ a.s. and $Z_T^{(n)} \to Z_T$ a.s. Use Question 1 to conclude that $\{Z_T^{(n)},\ n \ge 1\}$ is uniformly integrable. Recall further that for X, $X_n \in L^1$ $(n \ge 1)$, $X_n \to X$ in L^1 if and only if (i) $X_n \to X$ in probability and (ii) $\{X_n,\ n \ge 1\}$ is uniformly integrable.

Number of points can be found next to the questions; the grade will be calculated according to

$$\text{Grade} = \frac{\text{Number of points}}{10} + 1.$$

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