Exam Stochastic Differential Equations (SDE, Mastermath)

25-06-2018; 10:00 – 13:00. Total Points: 45.

Answer questions 1 – 3 and 4 – 5 on separate pages

No calculator/phone allowed

(2) 1. a. Is the product of two local martingales a local martingale? Prove or disprove the claim.

(2) b. Let $(M_t)_{t\geq 0}$ be a continuous local martingale. Suppose that for all $t \geq 0$, $\mathbb{E}\left(\sup_{0\leq s\leq t}|M_s|\right) < \infty$. Show that $(M_t)_{t\geq 0}$ is a real martingale.

(2) c. Let $(M_t)_{t\geq 0}$ be a positive local martingale s.t. $\mathbb{E}(M_0) < \infty$. Show that it is a supermartingale. Show further that it is a proper martingale if and only if for all $t \geq 0$ we have $\mathbb{E}(M_t) = \mathbb{E}(M_0)$.

2. Let B denote Brownian motion and let $a > 0$ be a constant. Furthermore define for $t \in [0, 1)$

$$M_t := \frac{1}{\sqrt{1-t}} \exp\left(-\frac{aB_t^2}{2(1-t)}\right).$$

(3) a. For which value, a_0, of a, is the above process a local martingale? Is it a martingale?

(3) b. Find a local martingale $(\varphi(s))_{s\geq 0}$ such that for $a = a_0$, the following holds:

$$M_t = \exp\left(\int_0^t \varphi(s)dB_s - \frac{1}{2}\int_0^t \varphi^2(s)ds\right).$$

(3) c. For general $a > 0$, show that $\lim_{t\to 1^-} M_t = 0$ a.s.
Suppose further that $r > 0$. Show that M_t^r converges to 0 in L_1 as t goes to 1^- iff $r \in (0, 1)$.

3. Let B be a Brownian motion. Consider a function $f \in L^2_{loc}(\mathbb{R}^+) = \{f : \mathbb{R}_+ \to \mathbb{R}, \forall K \text{ compact } : \int_K f(s)^2 ds < \infty\}$ and define $M = (M(t))_{t\geq 0}$ as:

$$M_t := \int_0^t f(s)dB_s.$$

(4) a. Show that M is a Gaussian process and compute the covariance. Deduce that M has independent increments.

(3) b. Assume that M is continuous a.s. Give a necessary and sufficient condition for f such that M is a Brownian motion.

(3) c. Let $g \in L^2_{loc}(\mathbb{R}_+)$ and we define $N = (N_t)_{t\geq 0}$

$$N_t := \int_0^t g(s)dB_s.$$

Show that the processes M and N are independent iff $f(t)g(t) = 0$ almost everywhere.

See next page.

4. Let (B_t) be a standard Brownian motion defined on a (filtered) probability space $(\Omega, \mathcal{F}\ (\mathcal{F}_t), P)$. Suppose X_t is a process satisfying the SDE

$$dX_t = X_t dB_t, \quad X_0 = 1. \qquad (*)$$

Define the process $(Z_t)_{t\in[0,1]}$ by

$$Z_t = X_t e^{-\int_0^t B_s^2 ds}.$$

(3) a. Apply Itô's formula to show that

$$dZ_t = Z_t(dB_t - B_t^2 dt), \quad Z_0 = 1. \qquad (**)$$

(2) b. Give the solution X_t of the SDE $(*)$.

(2) c. Use the solution in (b) to show that $\mathbb{E}(Z_t^2) \le e^t$.

(4) d. In considering the "integrated version" of the SDE $(**)$ on the interval $[0,1]$, two random variables are relevant: $\int_0^1 Z_t dB_t$ and $\int_0^1 Z_t B_t^2 dt$.

Use (c) to show that they satisfy the following properties.

$$(i)\ \mathbb{E}\left[\left(\int_0^1 Z_t dB_t\right)^2\right] < \infty \quad \text{and} \quad (ii)\ \mathbb{E}\left[\int_0^1 |Z_t B_t^2| dt\right] < \infty.$$

Hint: If needed, you may assume the expressions for the moments of Gaussian distribution, without proof. For example, $\mathbb{E}(\xi^4) = 3\sigma^4$ for $\xi \sim N(0, \sigma^2)$.

5. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, $\{B_t, t \ge 0\}$ a standard $(\mathcal{F}_t)$-Brownian motion.

(3) a. Suppose $\theta(\cdot)$ is a continuous (deterministic) function on $[0, \infty)$. Prove, from the definition of a martingale and the properties of stochastic integral, that the process M is a martingale on $[0, T]$, where

$$M_t = \exp\left(\int_0^t \theta(s) dB(s) - \frac{1}{2}\int_0^t \theta^2(s) ds\right).$$

[Do not use Itô formula in answering this question.]

(6) b. Suppose (ν_t), (μ_t) and (σ_t) are continuous adapted processes. Consider the stochastic process X_t satisfying

$$X_t = x + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dB_s, \quad 0 \le t \le T, \quad x \in \mathbb{R}.$$

Construct, under appropriate conditions on (ν_t), (μ_t) and (σ_t), a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), \mathbb{P})$ such that X has the following representation under $\mathbb{Q}$:

$$X_t = x + \int_0^t \nu_s\, ds + \int_0^t \sigma_s\, d\tilde{B}_s, \quad 0 \le t \le T,$$

where $(\tilde{B}_t)$ is a $\mathbb{Q}$-BM.

[Hint: Express X_t (under $\mathbb{P}$) in the desired form for some $\tilde{B}_t$.]

Number of points for each question can be found next to it; the grade will be calculated according to

$$\text{Grade} = \frac{\text{Number of points}}{45} \times 9 + 1.$$

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