Exam Vector Calculus
Module 3 Applied Physics and Applied Mathematics Bachelor

Codes 202001229-202001232

April 11, 2022, 8.45-10.45

The use of a book is not allowed

All answers must be justified and clearly formulated.

1. Consider a Cartesian coordinate system. Given the domain

$$D_1 := \{(x, y) \in \mathbb{R}^2 \; : \; x^2 + y^2 \leq 4\}.$$

The boundary of D_1 is denoted C_1. Given the domain

$$D_2 := \{(x, y) \in \mathbb{R}^2 \; : \; x^2 + y^2 \leq 1, \, y \geq 0\}.$$

The boundary of D_2 is denoted C_2.

We define the domain $D := D_1 \setminus D_2$. Hence D is the domain D_1 with the domain D_2 excluded. The boundary of the domain D is denoted $C = C_1 \cup C_2$.

The boundary C of the domain D has a positive orientation. The orientation of the boundaries C_1 and C_2 is defined by the orientation of the boundary C.

Given the vector field

$$\mathbf{F}(x, y) := 2y\mathbf{i} + xy^2\mathbf{j}.$$

a. Sketch the domain D. Indicate the Cartesian coordinate system and the proper orientation of the boundary C.

b. Give a parametrization of the curves C_1 and C_2.

c. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ without using Green's theorem.

d. Use Green's theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

2. The domain V is defined as

$$V := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 \le 1, x \ge 0, z \ge 0\}.$$

The boundary surface of the domain V is denoted as S. At the boundary surface S the external unit normal vector is given by $\hat{\mathbf{N}}$.

Given the vector field

$$\mathbf{F}(x, y, z) := xz\mathbf{i} + yz\mathbf{k}.$$

a. Sketch the surface S. Indicate the Cartesian coordinate system.

b. Calculate $\oiint_S \mathbf{F} \cdot \hat{\mathbf{N}} dS$ without using Gauss' theorem.

3. Given the oriented surface S_1, defined as

$$S_1 := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1, \ x < 0\}.$$

The boundary of the surface S_1 is denoted C_1 and has a positive orientation. The unit normal vector $\hat{\mathbf{N}}_1$ at S_1 has a negative x-component.

Given the oriented surface S_2, defined as

$$S_2 := \{(x, y, z) \in \mathbb{R}^3 \ : \ y^2 + z^2 \le 1, \ x = 0\}.$$

The boundary of the surface S_2 is denoted C_2 and has a positive orientation. The unit normal vector $\hat{\mathbf{N}}_2$ at S_2 has a positive x-component.

Given the surface $S := S_1 \cup S_2$ with unit normal vector $\hat{\mathbf{N}}$, where $\hat{\mathbf{N}} = \hat{\mathbf{N}}_1$ on surface S_1 and $\hat{\mathbf{N}} = \hat{\mathbf{N}}_2$ on surface S_2.

Given the vector field

$$\mathbf{F}(x, y, z) := x^2y\mathbf{i} + 4y^2z\mathbf{j} - y\mathbf{k}.$$

a. Sketch the surface S_1. Indicate the Cartesian coordinate system and the orientation of the boundary of S_1.

b. Calculate $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ without using Stokes' theorem.

c. Calculate $\iint_{S_2} \mathbf{curl}\, \mathbf{F} \cdot \hat{\mathbf{N}}_2 dS$.

d. Calculate $\iint_S \mathbf{curl}\, \mathbf{F} \cdot \hat{\mathbf{N}} dS$. Motivate your answer.

Grading

1: 11	2: 8	3: 8
1a: 1	2a: 1	3a: 1
1b: 2	2b: 7	3b: 4
1c: 4		3c: 2
1d: 4		3d: 1

total 27+3=30 points