E&M

UNIVERSITY OF TWENTE.

Boller, Offerhaus, Dhallé

E&M E-part, second installment

201800137 & 201700164

April 15 2019

Fields and electromagnetism

Instructions

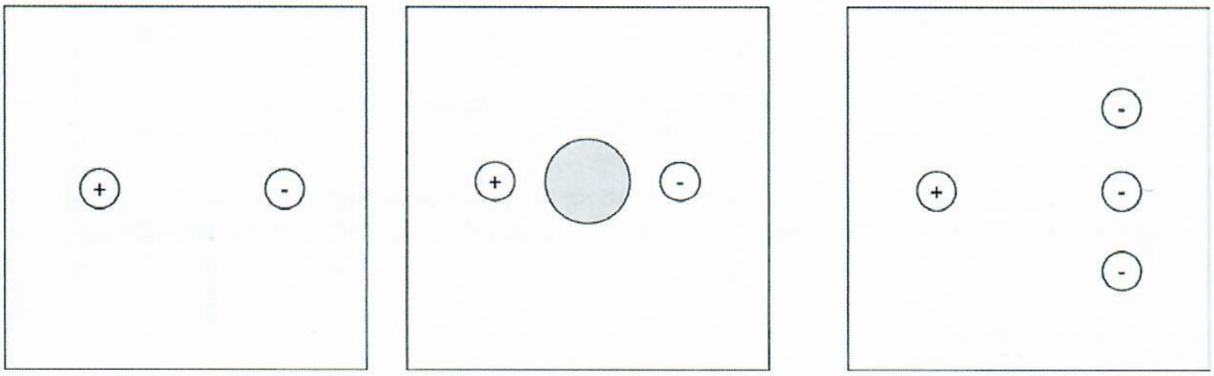
You have **2 hours** to complete the test. Clearly indicate your name and student number on every sheet that you hand in.

You may use a hand-written formula sheet containing maximum 10 equations. This sheet must be handed in together with your answers.

Before answering the questions, read all of them and start with the one you find easiest.

The amount of points to be obtained with each question is indicated next to the question number.

Problem 1 (15pts/100)



Above are three situations with various numbers of charges and a metal ball (grey sphere in the middle). Copy the 3 sketches and add the electric field lines appropriate for the situation. Take care to be consistent in the number of field-lines, indicate the direction of the field-lines and make sure that the scale of the drawing is such that we can judge the field close to the charges but also the field far away from the charges and at the middle distance (you can use multiple sketches if you prefer).

Problem 2 (20pts/100) Below you find eight statements. For each of them, indicate whether the statement is 'true' (T) or 'not true' (NT). Also include a brief argument why you agree or not (minimum 1 & maximum 5 lines per statement). Read the statements carefully, each word may be important!

a. A (closed) contour integral of the electric field is always zero.

b. A spherical shell carries a uniformly distributed positive surface charge. A positive point charge placed inside the shell will experience a force toward the center.

c. When one doubles the charge of (each of) two point charges, the force between them quadruples.

d. The electric flux trough an equipotential surface is always zero.

e. The charge distribution of a conducting sphere is always spherically symmetric, irrespective of its environment.

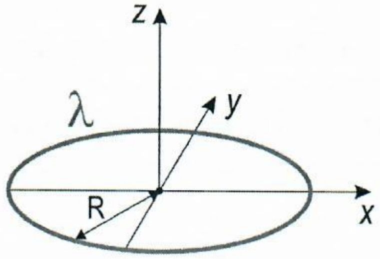
f. A dielectric that is placed in an inhomogeneous electric field will experience a force in the direction of the divergence of the field.

g. The potential difference between the plates of a disconnected capacitor decreases when one introduces a dielectric in between the plates.

h. A continuous **D** across an interface while **E** is discontinuous implies the presence of free charge.

Problem 3 (15pt/100)

1. Consider a homogenously charged ring with radius R and charge density λ (C/m), as below



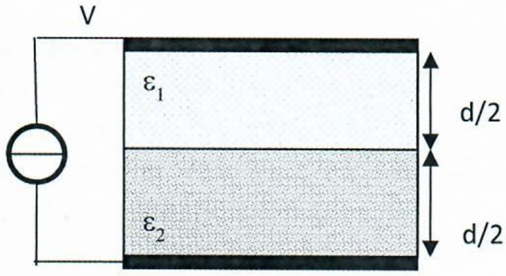
a. Show that the electric field **E** on the z-axis (at a perpendicular distance z above the midpoint of the circle) can be written as:

$$\mathbf{E}(z) = \frac{R\lambda}{2\varepsilon_0} \frac{z}{\left(R^2 + z^2\right)^{3/2}} \hat{\mathbf{e}}_z$$

b. Use this result to calculate the electric field along the z-axis resulting from a cylinder of radius R and length R centred on the z-axis around the origin with the axis of the cylinder aligned along the z-axis.

Problem 4 (25pt/100)

On the right is sideview of a capacitor filled with two different materials. The capacitor is charged to a voltage V_0 by a voltage source.



a. Calculate the free surface charge density σ_f for the case that the space between the capacitor plates is empty (no materials). Express your answer in ε_0, V_0 and d.
b. Calculate the free surface charge density σ_f for the case that the space between the capacitor plates is filled with the two different materials. Express your answer in ε_r, ε_0, V_0 and d.
c. Calculate the net bound surface charge density σ_b between the two materials. Express your answer in ε_r, ε_0, V_0 and d.

Problem 5 (25pt/100)

A sphere of radius R carries a charge Q , distributed homogeneously throughout its volume.

a. Calculate the total work U_{tot} that was needed to put this charge in the sphere.
b. Calculate the energy U_{inside} associated with the electric field inside the sphere.
c. Calculate the energy $U_{outside}$ associated with the electric field outside the sphere.

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

FUNDAMENTAL CONSTANTS

ϵ_0	$= 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$= 4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$= 3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$= 1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$= 9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Standaardintegralen .

$I = \int x^m (a^2+x^2)^n\,dx$ Noem $Y = \sqrt{a^2+x^2}$; $Y^2 = a^2+x^2$;

m	n	I	m	n	I
-2	-1/2	$-Y/(a^2x)$	1	-3/2	$-1/Y$
-2	-1	$-a^{-2}(\frac{1}{x}+\frac{1}{a}\arctan\frac{x}{a})$	1	-1	$\ln\lvert Y\rvert$
-1	-3/2	$a^{-2}(\frac{1}{Y}-\frac{1}{a}\ln\left\lvert\frac{a+Y}{x}\right\rvert)$	1	-1/2	Y
-1	-1/2	$-(1/a)\ln\lvert(a+Y)/x\rvert$	1	1/2	$\frac{1}{3}Y^3$
-1	-1	$a^{-2}\ln\lvert x/Y\rvert$	1	3/2	$\frac{1}{5}Y^5$
0	-3/2	$x/(a^2Y)$	2	-3/2	$\ln\lvert x+Y\rvert - x/Y$
0	-1	$a^{-1}\arctan(x/a)$	2	-1	$x - a\arctan(x/a)$
0	-1/2	$\ln\lvert x+Y\rvert$	2	-1/2	$\frac{1}{2}xY - \frac{1}{2}a^2\ln\lvert x+Y\rvert$
0	1/2	$\frac{1}{2}xY + \frac{1}{2}a^2\ln\lvert x+Y\rvert$	2	1/2	$\frac{1}{8}x(2x^2+a^2)Y - \frac{1}{8}a^4\ln\lvert x+Y\rvert$
0	3/2	$\frac{1}{8}x(2x^2+5a^2)Y + \frac{3}{8}a^4\ln\lvert x+Y\rvert$	3	-3/2	$Y + a^2/Y$
			3	-1/2	$\frac{1}{3}Y^3 - a^2Y$
			3	1/2	$\frac{1}{5}Y^5 - \frac{1}{3}a^2Y^3$

$I = \int \sin^m ax\cos^n ax\,dx$

m	n	I	m	n	I
1	0	$-(1/a)\cos ax$	1	1	$(\sin^2 ax)/2a$ of $-(\cos^2 ax)/2a$
0	1	$(1/a)\sin ax$	2	2	$-\frac{1}{32a}\sin 4ax + \frac{x}{8}$
1	-1	$-(1/a)\ln\lvert\cos ax\rvert$	1	n	$-\frac{\cos^{n+1}ax}{(n+1)a}$
-1	1	$(1/a)\ln\lvert\sin ax\rvert$	m	1	$\frac{\sin^{m+1}ax}{(m+1)a}$
2	0	$\frac{1}{2}x - \frac{1}{4a}\sin 2ax$	0	2	$\frac{1}{2}x + \frac{1}{4a}\sin 2ax$
3	0	$-\frac{1}{3a}\cos ax(\sin^2 ax + 2)$	0	3	$\frac{1}{3a}\sin ax(\cos^2 ax + 2)$
4	0	$\frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$	0	4	$\frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

6.5 Benaderingen voor $\lvert x\rvert \to 0$

$(1+x)^a$	$1+ax+\ldots$	$\sin x$	$x - x^3/6+\ldots$
e^x	$1+x+\ldots$	$\cos x$	$1 - x^2/2+\ldots$
$\ln(1+\mathrm{x})$	$x - x^3/2+\ldots$		