Offerhaus, Marpaung, Dhallé

E&M, M-part, 2nd instalment

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2017-201400535-2B
Fields and electromagnetism

Instructions You have **2 hours** to complete the test. Clearly indicate your name and student number on every sheet that you hand in. You may use a hand-written formula sheet containing maximum 10 equations. This sheet must be handed in together with your answers.

Before answering the questions, read all of them and start with the one you find easiest.

The amount of points to be obtained with each question is indicated next to the question number.

Problem 1 (10 points) Figure 1 gives a (simplified) representation of a quadrupole magnet that is used to focus a particle beam.

The larger circles at the four corners of the square with side *a* represent infinitely long straight line currents *I*, the top and bottom ones pointing out of the page (in the +*z* direction), the left and right ones into the page (-*z* direction).

The smaller circles at positions A (in the centre of the square), B (midway between the left and the top current), C (midway between A and the top current) and D (midway between the top and the right current) represent each time a proton (charge +q) coming out of the page with a velocity $\boldsymbol{v} = v_0\hat{\boldsymbol{z}}$.

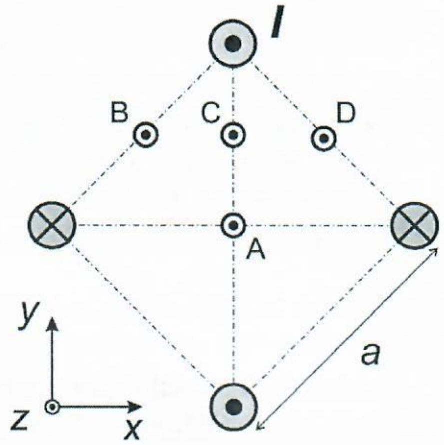


Figure 1: Quadrupole magnet and proton beam (Problem 1).

1.a Copy the figure and sketch the magnetic field vector **B** generated by the four line currents in each of the positions A, B, C and D. Pay attention to the direction and to the relative magnitude of **B** in the four positions.

1.b Once more copy the figure and in each of the four positions A-D, sketch the vectors representing the Lorentz force $\mathbf{F}_L$ acting on the protons.

Problem 2 (15 points) An infinite flat plate of thickness *d* carries a uniform current density *J* (Figure 2). Define a Cartesian coordinate system with the *yz*-plane as the plane of the plate and *z* the current direction.

2.a Derive an expression for the magnetic field **B** throughout space (i.e. for *x* < -d/2, for -d/2 < *x* < +d/2 and for +d/2 < *x*). Formulate your answer as a vector expression.

2.b Sketch a graph of the magnitude of the corresponding vector potential $|\mathbf{A}|$ as a function of *x*.

2.c What is the direction of **A**?

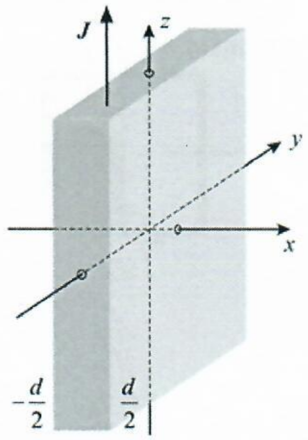


Figure 2: current-carrying plate (problem 2).

Problem 3 (20 points) Below you find eight statements. For each of them, indicate whether the statement is 'true' (T) or 'not true' (NT). Also include a brief argument why you agree or not (minimum 1 & maximum 5 lines per statement). Read the statements carefully, each word may be important!

T **3.a** The existence of magnetic monopoles would imply $\nabla.\mathbf{B} \neq 0$.

T **3.b** In vacuum, without electrical charges or currents, the curl of **B** equals zero.

?

3.c The magnetic field of the earth can be modelled as a dipole. Charged particles in the solar wind streaming from the sun towards the earth in the equatorial plane are deflected either to the north- or to the south pole of the earth, depending on the sign of their charge.

3.d Two current-carrying wires run parallel to each other. When one doubles the current in each of them, the force between them also doubles.

3.e Inside an infinitely long straight current-carrying solenoid, a paramagnetic material experiences a force along the magnetic field direction while a diamagnetic material feels a force directed against the field direction.

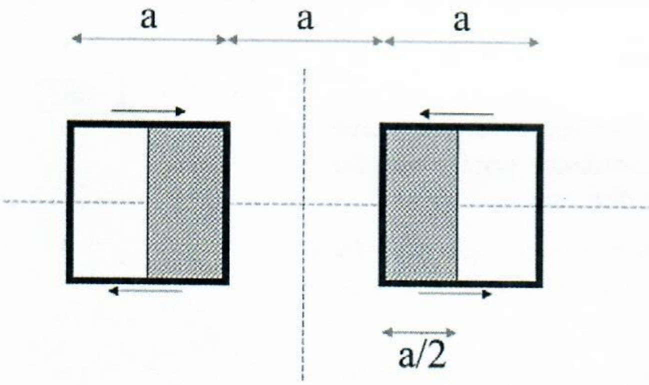
3.f Immediately outside a ferromagnet with high permeability μ_r, the magnetic **B**-field lines are nearly perpendicular to the surface of the magnet.

3.g Two cylindrical metal rods are driven vertically into the ground. They are separated a relatively large distance apart (compared to their diameter). The electrical resistance between the rods is then dominated by the electrical resistivity of the soil in the immediate vicinity of the rods.

Y **3.h** It requires work to move a closed conducting wire loop in an inhomogeneous magnetic field.

Problem 4 (20 points) Figure 3 schematically shows a thick toroidal coil with a square cross-section. The left picture is a side view of a vertical cut through the middle of the coil. The right picture is a top view of a horizontal cut through the middle. The thick black lines represent the outside of the coil and the black arrows next to the square cross-section indicate the direction of the current *I* that flows through the coil.

There are N turns in total in this coil. The inner diameter of the coil is a, the outer diameter is 3a. A material with a relative magnetic susceptibility μ_r=2 occupies the inner half of the inside of the coil (gray parts).



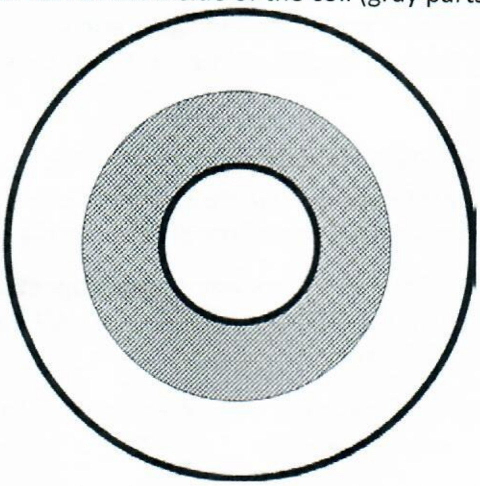


Figure 3: Toroidal coil partially filled with a magnetizable material (problem 4).

4.a Derive an expression for the magnetic field **H** (magnitude and direction) as a function of the radial distance *s* to the vertical center line and sketch the relevant component of **H** as a function of *s*.

4.b Derive an expression for the magnetic induction **B** as a function of *s* and sketch the relevant component of **B** in the graph that you created for part 4.a.

4.c Derive an expression for the magnetization **M** and from this, derive expressions for the surface current density K_B and the volume current density J_B. Explain the direction of these currents with respect to the current in the coil.

Problem 5 (15 points) Two long metal cylindrical tubes are placed coaxially (Figure 4). The inner one has radius a, the outer one radius $b > a$. The space in-between the tubes is filled with a material that has an electrical conductivity σ.

5.a The tubes are maintained at a potential difference V. Work out the electrical current that flows from one to the other over a length L.

5.b What is the electrical resistance R of this section of length L?

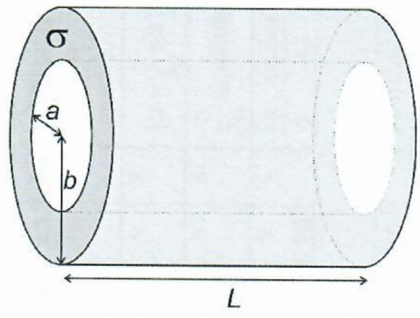


Figure 4: Radial conductor with cylindrical electrodes (Problem 5).

Problem 6 (20 points) Figure 5 shows a sketch of an infinite wire that carries a current I from left to right. Under the wire there is a rectangular pick-up loop with length $2a$ parallel to the wire and height a perpendicular to the wire, at a distance a from the wire.
The current I in the wire is described as: $I = I_0 \cos(\omega t)$.

6.a Calculate the magnetic induction B as a function of distance s to the wire.

6.b Suppose that the pick-up loop has an overall resistance of R. Calculate the induced current in the loop.

6.c Sketch in one graph the current in the main wire and the induced current in the top part of the pick-up loop. Make sure that the relative sign is sketched correctly.

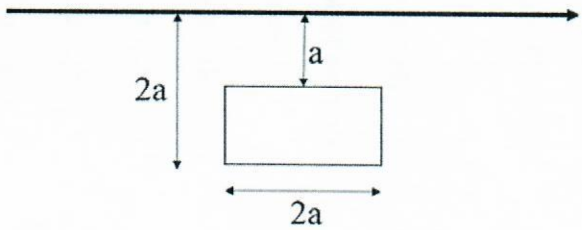


Figure 5: Current-carrying wire next to a rectangular conducting wire loop (Problem 6).

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\,d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

FUNDAMENTAL CONSTANTS

ϵ_0	$= 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$= 4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$= 3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$= 1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$= 9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Standaardintegralen .

$I = \int x^m (a^2+x^2)^n\,dx$ Noem $Y = \sqrt{a^2+x^2}$; $Y^2 = a^2+x^2$;

m	n	I	m	n	I
-2	-1/2	$-Y/(a^2x)$	1	-3/2	$-1/Y$
-2	-1	$-a^{-2}(\frac{1}{x}+\frac{1}{a}\arctan\frac{x}{a})$	1	-1	$\ln\lvert Y\rvert$
-1	-3/2	$a^{-2}(\frac{1}{Y}-\frac{1}{a}\ln\left\lvert\frac{a+Y}{x}\right\rvert)$	1	-1/2	Y
-1	-1/2	$-(1/a)\ln\lvert(a+Y)/x\rvert$	1	1/2	$\frac{1}{3}Y^3$
-1	-1	$a^{-2}\ln\lvert x/Y\rvert$	1	3/2	$\frac{1}{5}Y^5$
0	-3/2	$x/(a^2Y)$	2	-3/2	$\ln\lvert x+Y\rvert - x/Y$
0	-1	$a^{-1}\arctan(x/a)$	2	-1	$x - a\arctan(x/a)$
0	-1/2	$\ln\lvert x+Y\rvert$	2	-1/2	$\frac{1}{2}xY - \frac{1}{2}a^2\ln\lvert x+Y\rvert$
0	1/2	$\frac{1}{2}xY + \frac{1}{2}a^2\ln\lvert x+Y\rvert$	2	1/2	$\frac{1}{8}x(2x^2+a^2)Y - \frac{1}{8}a^4\ln\lvert x+Y\rvert$
0	3/2	$\frac{1}{8}x(2x^2+5a^2)Y + \frac{3}{8}a^4\ln\lvert x+Y\rvert$	3	-3/2	$Y + a^2/Y$
			3	-1/2	$\frac{1}{3}Y^3 - a^2Y$
			3	1/2	$\frac{1}{5}Y^5 - \frac{1}{3}a^2Y^3$

$I = \int \sin^m ax \cos^n ax\,dx$

m	n	I	m	n	I
1	0	$-(1/a)\cos ax$	1	1	$(\sin^2 ax)/2a$ of $-(\cos^2 ax)/2a$
0	1	$(1/a)\sin ax$	2	2	$-\frac{1}{32a}\sin 4ax + \frac{x}{8}$
1	-1	$-(1/a)\ln\lvert\cos ax\rvert$	1	n	$-\frac{\cos^{n+1} ax}{(n+1)a}$
-1	1	$(1/a)\ln\lvert\sin ax\rvert$	m	1	$\frac{\sin^{m+1} ax}{(m+1)a}$
2	0	$\frac{1}{2}x - \frac{1}{4a}\sin 2ax$	0	2	$\frac{1}{2}x + \frac{1}{4a}\sin 2ax$
3	0	$-\frac{1}{3a}\cos ax(\sin^2 ax + 2)$	0	3	$\frac{1}{3a}\sin ax(\cos^2 ax + 2)$
4	0	$\frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$	0	4	$\frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

6.5 Benaderingen voor $\lvert x\rvert \to 0$

$(1+x)^a$	$1+ax+\ldots$	$\sin x$	$x - x^3/6 + \ldots$
e^x	$1+x+\ldots$	$\cos x$	$1 - x^2/2 + \ldots$
$\ln(1+x)$	$x - x^3/2 + \ldots$		