Instructions

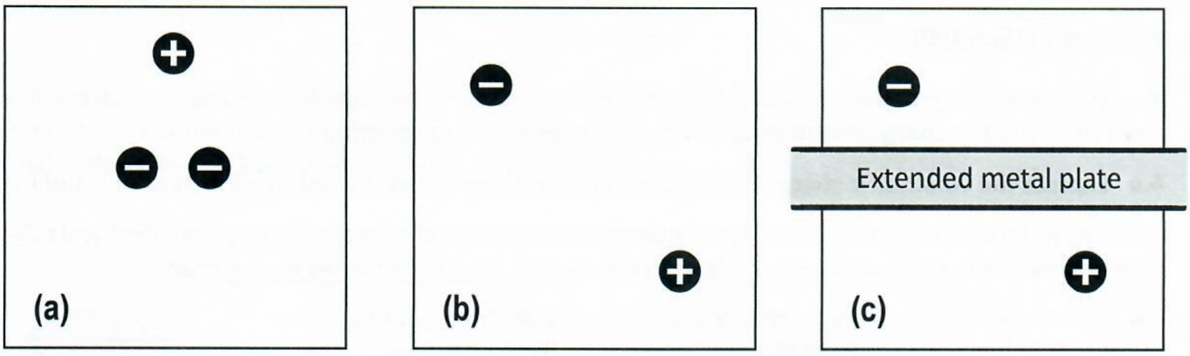
You have **2 hours** to complete the test. Clearly indicate your name and student number on every sheet that you hand in.

You may use a hand-written formula sheet containing maximum 10 equations. <u>This sheet must be handed in together with your answers</u>.

Before answering the questions, read all of them and start with the one you find easiest.

The amount of points to be obtained with each question is indicated next to the question number.

Problem 1 (15pts/100)



Above are three situations with various numbers of charges and a metal plate (the plate extends in the horizontal directions well beyond the boundaries of the sketch). Copy the 3 sketches and add the electric field lines appropriate for the situation.

Take care to be consistent in the number of field-lines, indicate their direction and make sure that the scale of the drawing is such that we can judge the field close to the charges but also the field far away from the charges and at the middle distance (you can use multiple sketches if you prefer).

Problem 2 (20pts/100)

2.a The Coulomb force between an electron and a proton that are separated by a distance *d* is stronger than the force between 2 electrons that are placed the same distance *d* apart;

2.b The curl of the gradient of a scalar field is always equal to zero;

2.c The total electric flux through a closed surface depends on the shape of the enclosed volume and on the amount of charge in that volume;

2.d When we double the strength of an electric field, we quadruple the energy stored in that field;

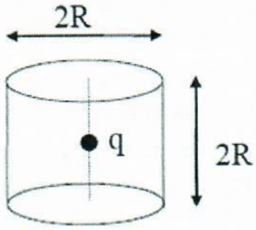
2.e Far away from a configuration of point charges, the equipotential surfaces corresponding to the electric field that is generated by those charges will be closed surfaces;

2.f Two capacitors with different capacitance C_1 and C_2 are charged with the same amount of charge. The capacitor with the lower C-value will have the lower voltage difference between its plates;

2.g Inside a dielectric material that is placed in an electric field, the field is lower than the external field;

2.h An **E**-field crosses an interface between free space ($\varepsilon_r = 1$) and a dielectric ($\varepsilon_r > 1$). The interface carries no free charge. Inside the dielectric, the E-field lines break away from the normal to the interface.

Problem 3 (15pt/100)



A positive point-charge q sits in the middle of a cylinder with height 2R and diameter 2R (see figure).

3.a Show that the electric flux through the vertical cylinder wall is $\Phi_E = \frac{q}{\sqrt{2}\varepsilon_0}$.

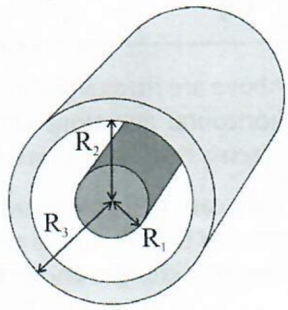
3.b Using this expression from question 3.a and Gauss' law, derive the flux through the circular top face of the cylinder.

Problem 4 (25pt/100)

An electrically insulating long cylindrical rod carries a charge of λ (in Coulomb per meter). The rod has a radius R_1 and the charge is distributed uniformly over its entire volume.

4.a Express the volumetric charge density ρ (in Coulomb per m^3) in the rod as a function of λ and R_1.

Concentrically around the insulating rod is placed a thick-walled electrically conducting cylindrical tube with an inner radius R_2 and an outer radius R_3 (see figure). This tube carries no net charge.



4.b Give a vector expression for the electric field **E** inside the insulating rod ($s < R_1$); in-between the rod and the conducting tube ($R_1 < s < R_2$); inside the wall of the tube ($R_2 < s < R_3$); and outside the tube ($R_3 < s$). Both rod and tube may be considered to be infinitely long. Express your answers in terms of s, ρ, R_1, R_2, R_3.

4.c Use the answer to question b. in order to sketch the magnitude of **E** as a function of the distance s to the central axis.

4.d What is the surface charge density σ (in Coulomb per m^2) on the inner surface of the tube (at $s = R_2$) and on its outer surface (at $s = R_3$).

4.e Choose the electric potential to be V = 0 in the middle of the wall of the conducting tube (at $s = \frac{R_2+R_3}{2}$). Derive the value of the potential on the central axis.

Continued on next page

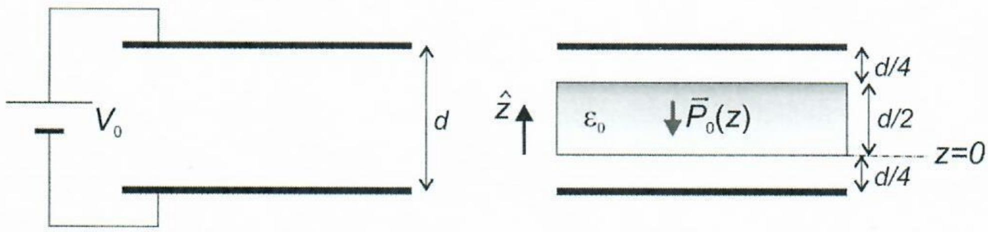
Problem 5 (25pt/100)

Below are two side-views of a parallel-plate capacitor that may be considered infinitely extended in the *x* and *y* directions, with its metal plates a distance *d* apart in the *z*-direction. In the left image, the capacitor is empty, with a voltage supply maintaining a potential difference V_0 between the plates.

On the right, the voltage supply is disconnected. After it was disconnected, a block of material was inserted in-between the plates. This block has a frozen-in non-uniform polarization that can be characterized as follows:

$$\vec{P}_0(z) = -\frac{\sigma_0 z}{d}\,\hat{z}$$

Here, $z = 0$ is taken to be the lower edge of the material and σ_0 is a constant. Also note that this polarization of the block is independent of external fields (i.e. apart from the frozen-in bound charge that is described by the equation above, the material has $\varepsilon_r = 1$).



5.a Derive an expression for the free surface charge density σ_f on the metal plates for the situation of the left, where σ_f is expressed in terms of *d* and V_0.

5.b Now the voltage source is decoupled and the block with a height of *d*/2 is inserted. Calculate the all the bound charges and make a sketch indicating where these bound charges reside.

5.c Calculate the electric field **E**(*z*) in the empty gaps between the plates and the material (i.e. for $-d/4 < z < 0$ and for $d/2 < z < 3d/4$). Express your answer in terms of *d*, V_0 and σ_0.

5.d Calculate the electric field **E**(*z*) inside the block (i.e. for $0 < z < d/2$). Express your answer in terms of *d*, V_0 and σ_0.

5.e Calculate the new potential difference *V* between the plates. Express your answer in terms of *d*, V_0 and σ_0.

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}};\quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}};\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}};\quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

FUNDAMENTAL CONSTANTS

ϵ_0	$= 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$= 4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
c	$= 3.00 \times 10^{8}\ \mathrm{m/s}$	(speed of light)
e	$= 1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
m	$= 9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

6.4 _Standaardintegralen .

$I = \int x^m (a^2+x^2)^n dx$ Noem $Y = \sqrt{a^2+x^2}$; $Y^2 = a^2+x^2$;

m	*n*	*I*	*m*	*n*	*I*
-2	-1/2	$-Y/(a^2x)$	1	-3/2	$-1/Y$
-2	-1	$-a^{-2}(\frac{1}{x}+\frac{1}{a}\arctan\frac{x}{a})$	1	-1	$\ln\lvert Y\rvert$
-1	-3/2	$a^{-2}(\frac{1}{Y}-\frac{1}{a}\ln\left\lvert\frac{a+Y}{x}\right\rvert)$	1	-1/2	Y
-1	-1/2	$-(1/a)\ln\lvert(a+Y)/x\rvert$	1	1/2	$\frac{1}{3}Y^3$
-1	-1	$a^{-2}\ln\lvert x/Y\rvert$	1	3/2	$\frac{1}{5}Y^5$
0	-3/2	$x/(a^2Y)$	2	-3/2	$\ln\lvert x+Y\rvert - x/Y$
0	-1	$a^{-1}\arctan(x/a)$	2	-1	$x - a\arctan(x/a)$
0	-1/2	$\ln\lvert x+Y\rvert$	2	-1/2	$\frac{1}{2}xY-\frac{1}{2}a^2\ln\lvert x+Y\rvert$
0	1/2	$\frac{1}{2}xY+\frac{1}{2}a^2\ln\lvert x+Y\rvert$	2	1/2	$\frac{1}{8}x(2x^2+a^2)Y-\frac{1}{8}a^4\ln\lvert x+Y\rvert$
0	3/2	$\frac{1}{8}x(2x^2+5a^2)Y+\frac{3}{8}a^4\ln\lvert x+Y\rvert$	3	-3/2	$Y+a^2/Y$
			3	-1/2	$\frac{1}{3}Y^3-a^2Y$
			3	1/2	$\frac{1}{5}Y^5-\frac{1}{3}a^2Y^3$

$I = \int \sin^m ax \cos^n ax\, dx$

m	*n*	*I*	*m*	*n*	*I*
1	0	$-(1/a)\cos ax$	1	1	$(\sin^2 ax)/2a$ of $-(\cos^2 ax)/2a$
0	1	$(1/a)\sin ax$	2	2	$-\frac{1}{32a}\sin 4ax+\frac{x}{8}$
1	-1	$-(1/a)\ln\lvert\cos ax\rvert$	1	n	$-\frac{\cos^{n+1} ax}{(n+1)a}$
-1	1	$(1/a)\ln\lvert\sin ax\rvert$	m	1	$\frac{\sin^{m+1} ax}{(m+1)a}$
2	0	$\frac{1}{2}x-\frac{1}{4a}\sin 2ax$	0	2	$\frac{1}{2}x+\frac{1}{4a}\sin 2ax$
3	0	$-\frac{1}{3a}\cos ax(\sin^2 ax+2)$	0	3	$\frac{1}{3a}\sin ax(\cos^2 ax+2)$
4	0	$\frac{3x}{8}-\frac{\sin 2ax}{4a}+\frac{\sin 4ax}{32a}$	0	4	$\frac{3x}{8}+\frac{\sin 2ax}{4a}+\frac{\sin 4ax}{32a}$

6.5 Benaderingen voor $\lvert x\rvert \to 0$

$(1+x)^a$	$1+ax+\ldots$	$\sin x$	$x-x^3/6+\ldots$
e^x	$1+x+\ldots$	$\cos x$	$1-x^2/2+\ldots$
$\ln(1+x)$	$x-x^3/2+\ldots$		