E&M Test 1
April 19th 2022

Boller, Offerhaus, Dhallé

201300164 & 201300183
Fields and electromagnetism

Instructions

You have **2 hours** to complete the test. Clearly indicate your name and student number on every sheet that you hand in.

You may use a hand-written formula sheet containing maximum 10 equations. This sheet must be handed in together with your answers.

Before answering the questions, read all of them and start with the one you find easiest.

The amount of points to be obtained with each question is indicated next to the question number.

Problem 1 (20pts/100)

An infinitely thin wire carries a uniform charge density $\lambda > 0$ and is depicted in figure 1. It has three sections. Section 1 and 3 can be considered to extend to infinity in the direction of the arrows. Section 2 is a quarter circle of radius R that connects these two straight segments. The point P is located at the center of this quarter circle.

1.a.) Calculate the **E**-field in the point P due to the semi-infinite line segment 3. Make sure to give both magnitude and direction of **E.**

1.b.) Calculate the **E**-field in point P due to the quarter-circle section 2

1.c.) Calculate the total **E**-field at P

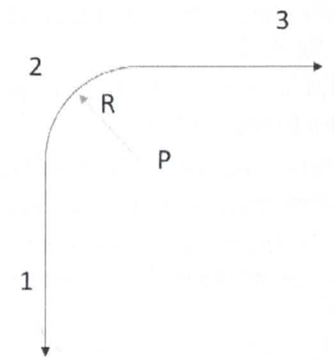


Figure 1 Infinitely long and -thin charged wire making a 90° bend.

Problem 2 (20pts/100)

Below you find 8 statements. For each of them, indicate whether the statement is 'true' **(T)** or 'not true' **(NT)**. Also include a brief argument why you agree or not (**minimum** 1 & **maximum** 5 lines per statement). Read the statements carefully, each word may be important!

2.a.) A positive point charge q is injected in-between two identical parallel planar and extended electrodes that both carry the same uniform surface charge $+\sigma$. In the absence of any other charges, the point charge experiences a force towards the center-plane exactly in the middle between the electrodes.

2.b.) The total flux of an **E**-field through any closed surface must be zero.

2.c.) Along a line that connects two identical point charges, the electric potential has a local minimum exactly in the middle between these charges.

2.d.) It requires four times as much energy to put a total charge 2q on a metal sphere of radius R than it does to put a total charge q on that same sphere.

2.e.) Inside an electrical conductor, the potential is always zero.

2.f.) When one pulls the charged plates of an extended parallel-plate capacitor apart (with the capacitor not attached to a voltage source), the voltage difference between the plates increases.

2.g.) Consider an electric field **E** that traverses a flat dielectric plate of thickness *d*. The plate is placed in vacuum. In the absence of free charges, the direction of the **E**-field outside the plate is the same on either side.

2.h.) A capacitor is connected to a voltage source that fixes the potential difference between its electrodes. When one introduces a dielectric in-between the electrodes (while keeping the source connected), the charge on the plates increases.

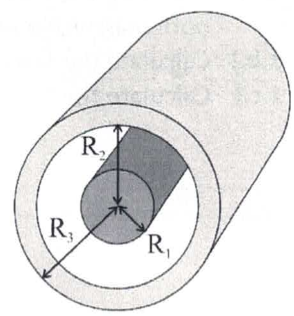
Problem 3 (25pts/100)

An electrically insulating long cylindrical rod carries a charge per unit lenght of λ (in Coulomb per meter). The rod has a radius R_1 and the charge is distributed uniformly over its entire volume.

3.a.) Express the *volumetric* charge density ρ (in Coulomb per m^3) in the rod as a function of λ and R_1.

Concentrically around the insulating rod is placed a thick-walled electrically conducting cylindrical tube with an inner radius R_2 and an outer radius R_3 (see figure). This tube carries no net charge.

3.b.) Give a *vector* expression for the electric field **E** inside the insulating rod ($s < R_1$); in-between the rod and the conducting tube ($R_1 < s < R_2$); inside the wall of the tube ($R_2 < s < R_3$); and outside the tube ($R_3 < s$). Both rod and tube may be considered to be infinitely long. Express your answers in terms of s, λ, R_1, R_2, R_3.



3.c.) Use the answer to question 3.b. in order to sketch the *magnitude* of **E** as a function of the distance *s* to the central axis.

3.d.) What is the surface charge density σ (in Coulomb per m^2) on the inner surface of the tube (at $s = R_2$) and on its outer surface (at $s = R_3$).

3.e.) Choose the electric potential to be V = 0 in the middle of the wall of the conducting tube (at $s = (R_2+R_3)/2$). Derive the value of the potential on the central axis.

Problem 4 (15pt/100)

Consider 4 concentric spherical shell with radius R, 2R, 3R and 4R that are covered with a charge +Q, +Q, -3Q and +Q. Calculate the potential at the center of the spheres with respect to infinity.

Problem 5 (20pt/100)

The space in between the plates of a planar capacitor is totally filled with two homogeneous flat layers of two insulating materials. The thickness of these layers is d_1 and d_2 (with $d_1 + d_2 = d$ the distance between the plates) and their relative permittivity ε_{r1} and ε_{r2}, with $\varepsilon_{r1} < \varepsilon_{r2}$. The potential difference between the plates is V.

5.a.) Make a sketch of the cross-section of the capacitor and in this sketch indicate the location of all charge (both free and bound).

5.b.) Sketch also the **E**- and **D**-field inside each layer, paying attention to the relative magnitude of the vectors in all regions.

5.c.) Calculate the magnitude of E and D in all regions (expressing it in terms of V, d_1, d_2, ε_{r1} and ε_{r2}).

5.d.) Calculate the net bound charge density at the interface between the two dielectric layers.

5.e.) Show that the capacitance of this capacitor is given by $C = A\varepsilon_0 \frac{\varepsilon_{r1}\varepsilon_{r2}}{\varepsilon_{r1}d_2+\varepsilon_{r2}d_1}$ with A the surface area of its plates.

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^8\ \mathrm{m/s}$	(speed of light)
$e = 1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

6.4 _Standaardintegralen .

$I = \int x^m (a^2+x^2)^n\,dx$ Noem $Y = \sqrt{a^2+x^2}$; $Y^2 = a^2+x^2$;

m	*n*	*I*	*m*	*n*	*I*
-2	-1/2	$-Y/(a^2x)$	1	-3/2	$-1/Y$
-2	-1	$-a^{-2}(\frac{1}{x}+\frac{1}{a}\arctan\frac{x}{a})$	1	-1	$\ln\|Y\|$
-1	-3/2	$a^{-2}(\frac{1}{Y}-\frac{1}{a}\ln\left\|\frac{a+Y}{x}\right\|)$	1	-1/2	Y
-1	-1/2	$-(1/a)\ln\|(a+Y)/x\|$	1	1/2	$\frac{1}{3}Y^3$
-1	-1	$a^{-2}\ln\|x/Y\|$	1	3/2	$\frac{1}{5}Y^5$
0	-3/2	$x/(a^2Y)$	2	-3/2	$\ln\|x+Y\|-x/Y$
0	-1	$a^{-1}\arctan(x/a)$	2	-1	$x-a\arctan(x/a)$
0	-1/2	$\ln\|x+Y\|$	2	-1/2	$\frac{1}{2}xY-\frac{1}{2}a^2\ln\|x+Y\|$
0	1/2	$\frac{1}{2}xY+\frac{1}{2}a^2\ln\|x+Y\|$	2	1/2	$\frac{1}{8}x(2x^2+a^2)Y-\frac{1}{8}a^4\ln\|x+Y\|$
0	3/2	$\frac{1}{8}x(2x^2+5a^2)Y+\frac{3}{8}a^4\ln\|x+Y\|$	3	-3/2	$Y+a^2/Y$
			3	-1/2	$\frac{1}{3}Y^3-a^2Y$
			3	1/2	$\frac{1}{5}Y^5-\frac{1}{3}a^2Y^3$

$I = \int \sin^m ax \cos^n ax\,dx$

m	*n*	*I*	*m*	*n*	*I*
1	0	$-(1/a)\cos ax$	1	1	$(\sin^2 ax)/2a$ of $-(\cos^2 ax)/2a$
0	1	$(1/a)\sin ax$	2	2	$-\frac{1}{32a}\sin 4ax+\frac{x}{8}$
1	-1	$-(1/a)\ln\|\cos ax\|$	1	n	$-\frac{\cos^{n+1} ax}{(n+1)a}$
-1	1	$(1/a)\ln\|\sin ax\|$	m	1	$\frac{\sin^{m+1} ax}{(m+1)a}$
2	0	$\frac{1}{2}x-\frac{1}{4a}\sin 2ax$	0	2	$\frac{1}{2}x+\frac{1}{4a}\sin 2ax$
3	0	$-\frac{1}{3a}\cos ax(\sin^2 ax+2)$	0	3	$\frac{1}{3a}\sin ax(\cos^2 ax+2)$
4	0	$\frac{3x}{8}-\frac{\sin 2ax}{4a}+\frac{\sin 4ax}{32a}$	0	4	$\frac{3x}{8}+\frac{\sin 2ax}{4a}+\frac{\sin 4ax}{32a}$

6.5 Benaderingen voor $|x| \to 0$

$(1+x)^a$	$1+ax+\ldots$	$\sin x$	$x-x^3/6+\ldots$
e^x	$1+x+\ldots$	$\cos x$	$1-x^2/2+\ldots$
$\ln(1+x)$	$x-x^3/2+\ldots$		