Test 2 for Probability Theory
(Module Signals and Uncertainty, 201300182)

Monday April 10, 2017, 8.45 - 11.15 uur

This test consists of 4 problems and one table. Motivate all answers. Using a calculator is *not* allowed.

1. The random variables X and Y are independent and have a $N(0, 1)$ distribution.

   a. Show by integration that the moment generating function (m.g.f.) of X is given by $\phi_X(t) = e^{t^2/2}$, and determine the m.g.f. of $X + Y$.

   b. Compute the correlation coefficient $\rho(X, X+Y)$. (If you don't know what this means, determine the covariance Cov$(X, X+Y)$ instead.)

2. In a school project, some children throw a fair dice 450 times, counting the number of times (N, say) that either a 5 or 6 comes up.

   a. Approximate the probability that N is larger than 160. Use a continuity correction and motivate your answer.

   In another project, the children also throw the dice, keeping track of two types of outcomes: either a 5 or 6 comes up (outcome A), or some other number 1, 2, 3 or 4 comes up (outcome B). The children keep throwing the dice until they have seen outcome A at least once *and* outcome B at least once.

   b. Determine the expected total number of trials.

   c. Determine (for the same experiment) the expected number of trials during which they saw outcome A.

3. The random variable Y has a probability density

$$f_Y(y) = \begin{cases} 5y^4 & \text{if } 0 < y < 1 \\ 0 & \text{else.} \end{cases}$$

   Given that $Y = y$ (with $0 < y < 1$), the random variable X has probability density

$$f_{X|Y}(x|y) = \begin{cases} 3x^2/y^3 & \text{if } 0 < x < y \\ 0 & \text{else.} \end{cases}$$

   a. Determine the joint density of X and Y.

   b. Determine $E(X|Y)$ and use this to find $E(X)$.

   c. Determine the conditional probability density of Y given $X = x$ for $0 < x < 1$.

P.T.O.

4. At time $t = 0$ two machines M_1 and M_2 start producing. Machine M_i breaks down after some time X_i and is then repaired for some time R_i, after which a second production time starts. When a machine breaks down for a second time, it is not repaired but taken out of use. We assume that X_1, X_2, R_1 and R_2 are independent, exponentially distributed random variables with expectations $E(X_1) = \frac{2}{3}$, $E(X_2) = \frac{2}{5}$ and $E(R_1) = E(R_2) = 1$.

   a. Give the distribution of the time T at which the first breakdown occurs. Also give $E(T)$.

   b. Determine the probability that M_1 is the first to break down, given that both machines are producing without interruptions up to time $t = 2$.

   c. Determine the probability that M_1 is in repair at time $t = 2$.

   d. Determine the probability that the first machine that breaks down is repaired by the time the other machine breaks down for the first time.

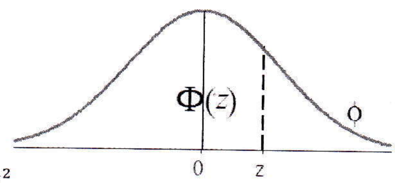
Norm:

1		2			3			4				Total
a	b	a	b	c	a	b	c	a	b	c	d	
4	2	3	3	3	2	4	4	3	2	3	3	36

Grade: $\frac{\text{total}}{4} + 1$ (this grade is 70% of the final grade for Probability Theory.)

Tab-1

Standard normal probabilities



The tabel gives the distribution function Φ for a N(0, 1) variable Z

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx$$

The last column gives the probability density of Z: $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

	Second decimal of z									
z	**0,00**	**0,01**	**0,02**	**0,03**	**0,04**	**0,05**	**0,06**	**0,07**	**0,08**	**0,09**	$\varphi(z)$
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359	0,3989
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753	0,3970
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141	0,3910
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517	0,3814
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879	0,3683
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224	0,3521
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549	0,3332
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852	0,3123
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133	0,2897
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389	0,2661
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621	0,2420
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830	0,2179
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015	0,1942
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177	0,1714
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319	0,1497
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441	0,1295
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545	0,1109
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633	0,0940
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706	0,0790
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767	0,0656
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817	0,0540
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857	0,0440
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890	0,0355
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916	0,0283
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936	0,0224
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952	0,0175
2,6	0,9953	0,9955	0,9956	0,9957	0,9959	0,9960	0,9961	0,9962	0,9963	0,9964	0,0136
2,7	0,9965	0,9966	0,9967	0,9968	0,9969	0,9970	0,9971	0,9972	0,9973	0,9974	0,0104
2,8	0,9974	0,9975	0,9976	0,9977	0,9977	0,9978	0,9979	0,9979	0,9980	0,9981	0,0079
2,9	0,9981	0,9982	0,9982	0,9983	0,9984	0,9984	0,9985	0,9985	0,9986	0,9986	0,0060
3,0	0,9987	0,9987	0,9987	0,9988	0,9988	0,9989	0,9989	0,9989	0,9990	0,9990	0,0044
3,1	0,9990	0,9991	0,9991	0,9991	0,9992	0,9992	0,9992	0,9992	0,9993	0,9993	0,0033
3,2	0,9993	0,9993	0,9994	0,9994	0,9994	0,9994	0,9994	0,9995	0,9995	0,9995	0,0024
3,3	0,9995	0,9995	0,9995	0,9996	0,9996	0,9996	0,9996	0,9996	0,9996	0,9997	0,0017
3,4	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9998	0,0012
3,5	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,0009
3,6	0,9998	0,9998	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,0006