Resit Test for Probability Theory
(Module Signals and Uncertainty, 201800138)
Friday July 3, 2020, 13.45 - 16.45 hours.

This test consists of 6 problems and 1 table (P.T.O.)
Use proper notation and motivate all answers.

Integrity statement

Please read the following paragraph carefully, copy the text below it verbatim to the first page of your work (handwritten) and sign it with your signature. If you fail to do so, your test will not be graded.
By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.
The only allowed sources for this test are:

- the book Introduction to Probability Models by Ross (hardcopy or pdf)
- the slides (printed or pdf)
- electronic devices, but only to be used:
  - for downloading the test and afterwards uploading your work to Canvas
  - to show the test/book/slides on your screen
  - to write the test (in case you prefer to use a tablet instead of paper to write on)

Text to be copied (handwritten) and signed:

I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.

Norm:

1			2			3				4				5				6				Total
a	b	c	a	b	c	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	
1	2	3	2	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	45

Grade: $\frac{\text{Total}}{5} + 1$

1. A card is chosen randomly from a deck of the standard 52 playing cards. Let E be the event that the selected card is a king and let F be the event that it is a heart.

   (a) Give the probility model for this experiment (i.e., both the sample space S and the probability measure P).

   (b) Prove or disprove that the events E and F are independent.

   (c) Show that the function P_E, given by $P_E(A) = P(A|E)$ (for any event $A \subset S$) is a probability measure on S.

2. The number of emails a certain professor gets per hour can be modeled by a Poisson distribution. On a weekday (Monday through Friday) the average is 10 emails/hour and during the weekend (Saturday and Sunday) the average is 3 emails/hour.

   (a) Calculate the probability of receiving more than 2 emails in a 1-hour interval on Sunday.

   (b) The professor really dislikes emails in a weekend during lunch time. Assuming that lunch time is a 1-hour period, approximate the probability that in total she receives more than 300 of such emails during a full year (assume a year has 104 weekend days). Use a continuity correction and motivate your answer.

   (c) A day is chosen at random, and a 1-hour interval is selected at random on the chosen day. What is the probability that the chosen day is a weekday if the professor received exactly 5 emails in that interval?

3. Suppose X and Y are independent Bernoulli random variables with success probability $p = 1/2$. Define $U = X + Y$ and $V = |X - Y|$.

   (a) Determine the joint probability function of U and V (e.g. in table form).

   (b) Determine the marginal probability functions of U and V.

   (c) Calculate $E(U)$ and $\mathrm{Var}(U)$.

   (d) Calculate the covariance between U and V.

4. Let F be the temperature on Manhattan at some random time of the year, measured in degrees Fahrenheit, and let C be the exact same temperature, but measured in degrees Celsius:

$$C = \frac{5}{9}F - \frac{160}{9}.$$

Let $\varphi(t)$ be the moment generating function of C, and assume $E(C) = 20$ and $\mathrm{Var}(C) = 500$.

(a) Determine $\frac{d^2}{dt^2}\varphi(t)|_{t=0}$.

(b) Determine $E(F)$ and $\mathrm{Var}(F)$.

(c) Determine $\mathrm{Cov}(F, C)$.

(d) Let ψ be the moment generating function of F. Express ψ in terms of φ.

5. The random variables X and Y have joint probability density function

$$f(x,y) = \begin{cases} 8xy, & 0 \le x \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine $P(Y \ge \frac{1}{2})$.

(b) Determine the marginal density of X.

(c) Determine $E\left(\frac{X}{Y}\right)$.

(d) Are X and Y independent?

6. Consider two machines. The life time of machine i is exponentially distributed with parameter $\lambda_i = i$ $(i = 1, 2)$. At time $t = 0$ both machines are operating. When a machine breaks down it is immediately replaced by a new, identical machine. All life times are independent.

(a) Determine the probability that machine 1 is the first machine that will break down.

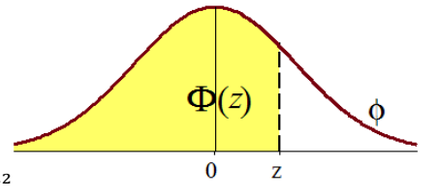
(b) When machine 2 is still working at time $t = 5$, what is the probability it will still be working at time $t = 7$?

Now assume that broken machines are not replaced (so after two breakdowns no machine is operating).

(c) Determine the probablity that by time t exactly one breakdown occurred.

(d) Determine the expected time between the first and the second breakdown.

Standard normal probabilities



The table gives the distribution function Φ for a N(0,1)-variable Z

$$\Phi(z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx$$

Last column: N(0,1)-density function (z in 1 dec.): $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

	Second decimal of **z**										
z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**φ(z)**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	0.3989
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	0.3970
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	0.3910
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	0.3814
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	0.3683
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	0.3521
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	0.3332
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	0.3123
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	0.2897
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	0.2661
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	0.2420
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	0.2179
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	0.1942
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	0.1714
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	0.1497
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	0.1295
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	0.1109
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	0.0940
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	0.0790
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	0.0656
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0.0540
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0.0440
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0.0355
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0.0283
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0.0224
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0.0175
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0.0136
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0.0104
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0.0079
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0.0060
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990	0.0044
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993	0.0033
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	0.0024
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0.0017
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.0012
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.0009
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.0006