Test 1 for Probability Theory
(Module Signals and Uncertainty, 202001342)
Monday March 6, 2023, 8.45 - 10.15 hour.

This test consists of 4 problems.
Use proper notation and motivate all answers.
Using a calculator is *not* allowed.

1. Consider a probability space (S, P).

   a. The first two axioms of Kolmogorov are given by (i) $P(E) \geq 0$ for any event $E \subset S$ and (ii) $P(S) = 1$. Give the third axiom (iii), and prove that (i)–(iii) imply that $P(E) \leq 1$ for any event $E \subset S$.

   b. Now assume S is a finite set and let P be given by the Laplace definition. Show that P satisfies (i) and (ii).

2. We choose a person at random from the Dutch population and measure the length X of the person. Assume the probabilities of the person being male or female are both 0.5. The length of a male person is assumed to be normally distributed with expectation μ and standard deviation σ, while the length of a woman is normally distributed with expectation ν and standard deviation τ. (Hint: define a suitable event to indicate whether the chosen person is male or female.)

   a. Give an expression for $P(X \leq x)$ in terms of Φ and the given parameters, where Φ is the distribution function of a standard normal random variable Z.

   b. Give an expression (again in terms of Φ and the given parameters) for the probability that a woman was chosen if we already know that the length of the person was not more than x.

   c. What is the distribution of $2Y - 10$ where Y is the weight of a male person?

3. A second-hand car dealer has 20 cars for sale, five of which are reliable (and 15 not). We pick two cars at random for a test drive. Let X be the number of reliable cars in our pick.

   a. Give the range S_X and the probability mass function of X.

   b. Sketch the form of the distribution function of X, with relevant values on the axes. (If you did not answer a. above, then assume that X takes values 0, 1 and 2 with probabilities 21/39, 16/39 and 2/39 respectively).

   c. Can the distribution of X be reasonably approximated by a Binomial distribution with $n = 2$ and $p = 1/4$? Argue why (not); no computations are needed.

4. The random variable X has an exponential distribution with parameter 3.

   a. Determine the probability $P(6X > X^2 + 8)$.

   b. Determine the density of the random variable Y given by $Y = e^{-X}$.

   c. Determine $E[e^{-X}]$.

Norm: (grade = total/3 + 1)

1		2			3			4			Total
a	b	a	b	c	a	b	c	a	b	c	
3	2	3	2	2	3	2	2	3	3	2	27