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Signals & Transforms — TEST 1
(part of AM module 4 — 201800138)

Date:	13-05-2019
Place:	Sports Center (SC1)
Time:	8:45–10:15 (till 10:45 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	NONE

The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.

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1. Let $\beta \in \mathbb{R}$, and consider the complex function $f(t)$ with period $T = 1$ that equals

$$f(t) = e^{i2\pi\beta t} \qquad \text{for } t \in [0,1).$$

   (a) Show that the complex Fourier coefficients f_k equal

$$f_k = e^{i\pi(\beta-k)} \operatorname{sinc}(\pi(\beta-k)).$$

   (b) For which $\beta, t \in \mathbb{R}$ does the Fourier series equal $f(t)$.

   (c) Determine the generalized derivative of $f(t)$ for all $t \in \mathbb{R}$.

   (d) Suppose β is not an integer. Use Parseval to compute $\sum_{k=-\infty}^{\infty} \frac{1}{(\beta-k)^2}$.

2. Determine the convolution of $f(t) = \mathbb{1}(t+1) + \delta(t+2)$ and $g(t) = e^{-t}\mathbb{1}(t-1)$.

3. What is the definition of a *Hilbert space*?

4. Let $\mathscr{C}([-2,2];\mathbb{R})$ denote the normed vector space of continuous functions $f : [-2,2] \to \mathbb{R}$ with norm $\|f\|_1 = \int_{-2}^{2} |f(t)|\,dt$.

   (a) Show that this is indeed a norm on $\mathscr{C}([-2,2];\mathbb{R})$.

   (b) Let A be the linear mapping $A : \mathscr{C}([-2,2];\mathbb{R}) \to \mathbb{R}$ defined as

$$A(f) = \int_{-2}^{2} (t^2-1) f(t)\,dt.$$

   On $\mathscr{C}([-2,2];\mathbb{R})$ we use as norm $\|f\|_1$, and on $\mathbb{R}$ we use as norm the absolute value. Determine the operator norm $\|A\|$.

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problem:	1	2	3	4
points:	5+2+2+2	6	2	4+4

Test grade is $1 + 9p/p_{\max}$

Property		Condition
Sifting	$\int_{-\infty}^{\infty} \delta(t-b) f(t)\, \mathrm{d}t = f(b)$	$f(t)$ continuous at $t = b$
-	$f(t)\delta(t-b) = f(b)\delta(t-b)$	$f(t)$ continuous at $t = b$
Convolution	$(f * \delta)(t) = f(t)$	
Scaling	$\delta(at-b) = \frac{1}{\lvert a\rvert}\delta(t-\frac{b}{a})$	
-	$\int_{-\infty}^{t} \delta(\tau)\, \mathrm{d}\tau = \mathbb{1}(t)$	$t \neq 0$

Property	Time domain: $f(t)$	Frequency domain: f_k
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha f_k + \beta g_k$
Time-shift	$f(t-\tau)$, $(\tau \in \mathbb{R})$	$\mathrm{e}^{-\mathrm{i}k\omega_0\tau} f_k$
Time-reversal	$f(-t)$	f_{-k}
Conjugation	$f^*(t)$	f_{-k}^*
Frequency-shift	$\mathrm{e}^{\mathrm{i}n\omega_0 t} f(t)$, $(n \in \mathbb{Z})$	f_{k-n}